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## Global Dynamics of Some Discrete Dynamical Systems With Applications

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GLOBAL DYNAMICS OF SOME DISCRETE DYNAMICAL SYSTEMS WITH  
APPLICATIONS

BY

ARZU BILGIN

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF  
DOCTOR OF PHILOSOPHY  
IN  
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2016

DOCTOR OF PHILOSOPHY DISSERTATION  
OF  
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## ABSTRACT

In my first manuscript, I investigate the global character of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \quad (1)$$

with several period-two solutions, where  $f$  is increasing in all its variables. I show that the boundaries of the basins of attractions of different locally asymptotically stable equilibrium solutions or period-two solutions are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium solutions or period-two solutions. An application of my results give global dynamics of three feasible models in population dynamics which includes the nonlinearity of Beverton-Holt and sigmoid Beverton-Holt types.

In this paper I consider Eq.(1) which has three equilibrium points and up to three minimal period-two solutions which are in North-East ordering. More precisely, I will give sufficient conditions for the precise description of the basins of attraction of different equilibrium points and period-two solutions. The results can be immediately extended to the case of any number of the equilibrium points and the period-two solutions by replicating my main results.

In my second manuscript, I investigate the asymptotic behavior of the solutions of the system of difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, \dots, \vec{x}_{n-k}), \quad n = 0, 1, \dots,$$

where  $k \in \{0, 1, \dots\}$  and the initial conditions are real vectors. I give some effective conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation. My results are based on application of the linearization technique. I illustrate my results with many examples that include some transition functions from mathematical biology such as linear (also known as Holling type I functions) [5], Beverton-Holt (also known as Holling type II functions or Holling hyperbolic functions), sigmoid Beverton-Holt (also known as Holling type III functions or sigmoid functions) and exponential functions. In this paper I extend some of the results from [4] to the case of vector equation (21).

In my third manuscript, I consider the cooperative system

$$x_{n+1} = ax_n + \frac{by_n^2}{1 + y_n^2} \quad y_{n+1} = \frac{cx_n^2}{1 + x_n^2} + dy_n, \quad n = 0, 1, \dots,$$

where all parameters  $a, b, c, d$  are positive numbers and the initial conditions  $x_0, y_0$  are nonnegative numbers. I describe the global dynamics of this system in number of cases. An interesting feature of this system is that exhibits a coexistence of locally stable equilibrium and locally stable periodic solution as well as the Allee's effect. All global dynamic results for this system can be extended to the general cooperative discrete system in the plane.

In my fourth manuscript, I present some basic discrete models in populations dynamics of single species with several age classes. Starting with the basic Beverton-Holt model that describes the change of single species I discuss its basic properties such as a convergence of all solutions to the equilibrium, oscillation of solutions about the equilibrium solutions, Allee's effect, etc. I consider the effect of the constant and periodic immigration and emigration on the global properties of Beverton-Holt model. I also consider the effect of the periodic environment on the global properties of Beverton-Holt model. In this paper I extend Theorems 34 -39 to the case of several generation model with special emphasis on three generation model. I prove general results about asymptotic stability. both local and global which cover all kind of transition or response functions such as linear , Beverton-Holt, sigmoid Beverton-Holt and exponential functions. In order to do so, I introduce some tools in Section 2 which contains some global attractivity results for monotone systems and some difference inequalities results which lead to precise global attractivity results for non-autonomous asymptotically autonomous difference equations. In Sections 3 and 4 I obtain fairly general results for local and global asymptotic stability of  $k$ -th generations model that extend all results in this section. In the special case of three generation model I find the precise basins of attraction of all locally stable equilibrium solutions and locally stable period-two solutions.

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## DEDICATION

To Dr. Kulenović,  
to my soulmate Mehmet, my beloved son Ahmet Yasin,  
and to my parents.



## **PREFACE**

This is a dissertation in Manuscript format

Manuscript 1 of this thesis was published as:

A. Bilgin, E. Pilav and M. R. S. Kulenović, Basins of Attraction of Period-Two  
Solutions of Monotone Difference Equations, Advances in Difference Equations,  
2016:2016, 25 p.

Manuscript 2 of this thesis was published as:

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Manuscript 3 of this thesis was submitted as:

A. Bilgin, A. Brett, E. Pilav and M. R. S. Kulenović, Global Dynamics of  
Cooperative Discrete System in the Plane.

Manuscript 4 of this thesis was submitted as:

A. Bilgin and M. R. S. Kulenović, Global Asymptotic Stability for Discrete  
Single Species Population Models.

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MANUSCRIPT 1

**Basins of Attraction of Period-Two Solutions of Monotone Difference  
Equations**

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## 1.1 Introduction

Let  $I$  be some interval of real numbers and let  $f \in C^1[I \times I, I]$  be such that  $f(I \times I) \subseteq \mathcal{K}$  where  $\mathcal{K} \subseteq I$  is a compact set. Let  $\bar{x}_0, \bar{x}_{SW}, \bar{x}_{NE} \in I$ ,  $\bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  be three equilibrium points of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \quad (2)$$

where  $f$  is a continuous and increasing function in both variables. There are several global attractivity results for Eq.(2) which give the sufficient conditions for all solutions to approach a unique equilibrium. I list three such results:

The first theorem, which has also been very useful in applications to mathematical biology, was really motivated by a problem in [7].

**Theorem 1** (See [7] and [54], p. 9). *Let  $I \subseteq [0, \infty)$  be some interval and assume that  $f \in C[I \times I, (0, \infty)]$  satisfies the following conditions:*

- (i)  $f(x, y)$  is non-decreasing in each of its arguments;
- (ii) Eq.(2) has a unique positive equilibrium point  $\bar{x}$  and the function  $f(x, x)$

*satisfies the **negative feedback condition***

$$(x - \bar{x})(f(x, x) - x) < 0 \quad \text{for every } x \in I - \{\bar{x}\}. \quad (3)$$

*Then every positive solution of Eq.(2) with initial conditions in  $I$  converges to  $\bar{x}$ .*

**Theorem 2** *Let  $[a, b]$  be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

*is a continuous function satisfying the following properties:*

- (a)  $f(x, y)$  is non-decreasing in each of its arguments;
- (b) Eq.(2) has a unique equilibrium  $\bar{x} \in [a, b]$ .

*Then every solution of Eq.(2) converges to  $\bar{x}$ .*

Theorems 1 and 2 are extended to  $k$ -th order difference equation where the right hand side is non-decreasing function of all its variables, see [7, 54].

The following result has been obtained in [1].

**Theorem 3** *Let  $I \subseteq \mathbb{R}$  and let  $f \in C[I \times I, I]$  be a function which increases in both variables. Then for every solution of Eq.(2) the subsequences  $\{x_{2n}\}_{n=0}^{\infty}$  and  $\{x_{2n+1}\}_{n=-1}^{\infty}$  of even and odd terms of the solution do exactly one of the following:*

- (i) *Eventually they are both monotonically increasing.*
- (ii) *Eventually they are both monotonically decreasing.*
- (iii) *One of them is monotonically increasing and the other is monotonically decreasing .*

**Remark 1** Theorem 1 is actually a special case of Theorem 2. Indeed (3) implies that there exist  $a$  and  $b, a < \bar{x} < b$  such that  $f(a, a) > a, f(b, b) < b$ , which in view of monotonicity of  $f$  implies that

$$f : [a, b] \times [a, b] \rightarrow [a, b],$$

and so all conditions of Theorem 2 are satisfied. Furthermore, Theorem 2 is a special case of Theorem 3 if I additionally assume non-existence of period-two solutions and singular points on the boundary of the region which may be the limiting points of the solutions of equation (2). None of these results provide any information about the basins of attraction of different equilibrium solutions or period-two solutions when there exist several equilibrium solutions and period-two solutions. Theorem 3 has been applied to many equations, see [1, 3] and references therein but so far no example of equation (2) with a function  $f$  increasing in both variables, which has stable equilibrium points and stable period-two solutions has



been exhibited. In this paper I give several such examples for difference equations of the type

$$x_{n+1} = f_0(x_n) + f_1(x_{n-1}), \quad n = 0, 1, \dots \quad (4)$$

where  $f_i, i = 0, 1$  are continuous and increasing functions. Many difference equations in mathematical biology are of the form (4), see [74] and references therein. As I have shown in [17] if first order difference equation  $x_{n+1} = f_1(x_n)$  has 2 equilibrium points the corresponding delay difference equation  $x_{n+1} = f_1(x_{n-1})$  has one period-two solution. Now if  $f_1$  dominates  $f_0$  in equation (4) then one may expect the equation (4) will have period-two solution as well, which I show in this paper. Similarly if difference equation  $x_{n+1} = f_1(x_n)$  has  $k$  equilibrium points the corresponding delay difference equation  $x_{n+1} = f_1(x_{n-1})$  has  $k(k-1)/2$  period-two solutions, and if  $f_1$  dominates  $f_0$  in equation (4) then I show that equation (4) has period-two solution as well.

In [3] authors consider difference equation (2) with several equilibrium points under the condition of nonexistence of minimal period-two solutions and determine the basins of attraction of different equilibrium solutions. In this paper I consider Eq.(2) which has three equilibrium points and up to three minimal period-two solutions which are in North-East ordering. More precisely, I will give sufficient conditions for the precise description of the basins of attraction of different equilibrium points and period-two solutions. The results can be immediately extended to the case of any number of the equilibrium points and the period-two solutions by replicating our main results. An application of our results gives precise description of the basin of attraction of all attractors of several difference equations, which are feasible models in population dynamics. Precisely, I illustrate our results with applications to three difference equations where all functions are linear, Beverton-Holt or sigmoid Beverton-Holt. In fact, our general results here are motivated by

equations (9) and (148). Our results give first examples of difference equations with coexisting stable equilibrium solutions and stable period-two solutions.

## 1.2 Preliminaries

I now give some basic notions about monotone maps in the plane and connection between equation (2) and the monotone map.

Consider a map  $T$  on a nonempty set  $\mathcal{S} \subset \mathbb{R}^2$ , and let  $\bar{\mathbf{e}} \in \mathcal{S}$ . The point  $\bar{\mathbf{e}} \in \mathcal{S}$  is called a *fixed point* if  $T(\bar{\mathbf{e}}) = \bar{\mathbf{e}}$ . An *isolated* fixed point is a fixed point that has a neighborhood with no other fixed points in it. A fixed point  $\bar{\mathbf{e}} \in \mathcal{S}$  is an *attractor* if there exists a neighborhood  $\mathcal{U}$  of  $\bar{\mathbf{e}}$  such that  $T^n(\mathbf{x}) \rightarrow \bar{\mathbf{e}}$  as  $n \rightarrow \infty$  for  $\mathbf{x} \in \mathcal{U}$ ; the *basin of attraction* is the set of all  $\mathbf{x} \in \mathcal{S}$  such that  $T^n(\mathbf{x}) \rightarrow \bar{\mathbf{e}}$  as  $n \rightarrow \infty$ . A fixed point  $\bar{\mathbf{e}}$  is a global attractor on a set  $\mathcal{K}$  if  $\bar{\mathbf{e}}$  is an attractor and  $\mathcal{K}$  is a subset of the basin of attraction of  $\bar{\mathbf{e}}$ . If  $T$  is differentiable at a fixed point  $\bar{\mathbf{e}}$ , and if the Jacobian  $J_T(\bar{\mathbf{e}})$  has one eigenvalue with modulus less than one and a second eigenvalue with modulus greater than one,  $\bar{\mathbf{e}}$  is said to be a *saddle*. If one of the eigenvalues of  $T$  has absolute values 1 and a second eigenvalue has modulus greater (resp. less) than one, then  $\bar{\mathbf{e}}$  is said to be a *nonhyperbolic* of *stable* (resp. *unstable*) type. See [20] for additional definitions (stable and unstable manifolds, asymptotic stability).

Consider a partial ordering  $\preceq$  on  $\mathbb{R}^2$ . Two points  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$  are said to be *related* if  $\mathbf{v} \preceq \mathbf{w}$  or  $\mathbf{w} \preceq \mathbf{v}$ . Also, a strict inequality between points may be defined as  $\mathbf{v} \prec \mathbf{w}$  if  $\mathbf{v} \preceq \mathbf{w}$  and  $\mathbf{v} \neq \mathbf{w}$ . A stronger inequality may be defined as  $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$  if  $\mathbf{v} \preceq \mathbf{w}$  with  $v_1 \neq w_1$  and  $v_2 \neq w_2$ . For  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^2$ , the *order interval*  $[\![\mathbf{u}, \mathbf{v}]\!]$  is the set of all  $\mathbf{x} \in \mathbb{R}^2$  such that  $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$ .

A map  $T$  on a nonempty set  $\mathcal{S} \subset \mathbb{R}^2$  is a continuous function  $T : \mathcal{S} \rightarrow \mathcal{S}$ . The map  $T$  is *monotone* if  $\mathbf{v} \preceq \mathbf{w}$  implies  $T(\mathbf{v}) \preceq T(\mathbf{w})$  for all  $\mathbf{v}, \mathbf{w} \in \mathcal{S}$ , and it is *strongly monotone* on  $\mathcal{S}$  if  $\mathbf{v} \prec \mathbf{w}$  implies that  $T(\mathbf{v}) \ll T(\mathbf{w})$  for all  $\mathbf{v}, \mathbf{w} \in \mathcal{S}$ . The

map is *strictly monotone* on  $\mathcal{S}$  if  $\mathbf{v} \prec \mathbf{w}$  implies that  $T(\mathbf{v}) \prec T(\mathbf{w})$  for all  $\mathbf{v}, \mathbf{w} \in \mathcal{S}$ . Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper I shall use the *North-East ordering* for which the positive cone is the first quadrant, i.e. this partial ordering is defined by

$$(x_1, y_1) \preceq_{ne} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \quad \text{and} \quad y_1 \leq y_2 \quad (5)$$

A map  $T$  on a nonempty set  $\mathcal{S} \subset \mathbb{R}^2$  which is monotone with respect to the North-East ordering is called cooperative and a map monotone with respect to the South-East ordering

$$(x_1, y_1) \preceq_{se} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \quad \text{and} \quad y_1 \geq y_2 \quad (6)$$

is called competitive.

If  $T$  is differentiable map on a nonempty set  $\mathcal{S}$ , a sufficient condition for  $T$  to be strongly monotonic with respect to the NE ordering is that the Jacobian matrix at all points  $\mathbf{x}$  has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad (7)$$

provided that  $\mathcal{S}$  is open and convex.

For  $\mathbf{x} \in \mathbb{R}^2$ , define  $Q_\ell(\mathbf{x})$  for  $\ell = 1, \dots, 4$  to be the usual four quadrants based at  $\mathbf{x}$  and numbered in a counterclockwise direction, for example,  $Q_1(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 : x_1 \leq y_1, x_2 \leq y_2\}$ . The (open) ball of radius  $r$  centered at  $\mathbf{x}$  is denoted with  $\mathcal{B}(\mathbf{x}, r)$ . If  $\mathcal{K} \subset \mathbb{R}^2$  and  $r > 0$ , write  $\mathcal{K} + \mathcal{B}(\mathbf{0}, r) := \{\mathbf{x} : \mathbf{x} = \mathbf{k} + \mathbf{y} \text{ for some } \mathbf{k} \in \mathcal{K} \text{ and } \mathbf{y} \in \mathcal{B}(\mathbf{0}, r)\}$ . If  $\mathbf{x} \in [-\infty, \infty]^2$  is such that  $\mathbf{x} \preceq \mathbf{y}$  for every  $\mathbf{y}$  in a set  $\mathcal{Y}$ , I write  $\mathbf{x} \preceq \mathcal{Y}$ . The inequality  $\mathcal{Y} \preceq \mathbf{x}$  is defined similarly.

The next result in [67] is stated for order-preserving maps on  $\mathbb{R}^n$  and it also holds in ordered Banach spaces [5].

**Theorem 4** *For a nonempty set  $R \subset \mathbb{R}^n$  and  $\preceq$  a partial order on  $\mathbb{R}^n$ , let  $T : R \rightarrow R$  be an order preserving map, and let  $a, b \in R$  be such that  $a \prec b$  and  $\llbracket a, b \rrbracket \subset R$ . If  $a \preceq T(a)$  and  $T(b) \preceq b$ , then  $\llbracket a, b \rrbracket$  is an invariant set and*

- i. *There exists a fixed point of  $T$  in  $\llbracket a, b \rrbracket$ .*
- ii. *If  $T$  is strongly order preserving, then there exists a fixed point in  $\llbracket a, b \rrbracket$  which is stable relative to  $\llbracket a, b \rrbracket$ .*
- iii. *If there is only one fixed point in  $\llbracket a, b \rrbracket$ , then it is a global attractor in  $\llbracket a, b \rrbracket$  and therefore asymptotically stable relative to  $\llbracket a, b \rrbracket$ .*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [5, 67].

**Corollary 1** *If the nonnegative cone of a partial ordering  $\preceq$  is a generalized quadrant in  $\mathbb{R}^n$ , and if  $T$  has no fixed points in  $\llbracket u_1, u_2 \rrbracket$  other than  $u_1$  and  $u_2$ , then the interior of  $\llbracket u_1, u_2 \rrbracket$  is either a subset of the basin of attraction of  $u_1$  or a subset of the basin of attraction of  $u_2$ .*

I say that  $f$  is *strongly increasing* in both arguments if it is increasing, differentiable and have both partial derivatives positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Eq.(2) follows from the fact that if  $f$  is strongly increasing, then a map associated to Eq.(2) is a cooperative map on  $\mathcal{I} \times \mathcal{I}$  while the second iterate of a map associated to Eq.(2) is a strictly cooperative map on  $\mathcal{I} \times \mathcal{I}$ .

Set  $x_{n-1} = u_n$  and  $x_n = v_n$  in Eq.(2) to obtain the equivalent system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= f(v_n, u_n) \end{aligned} \quad , \quad n = 0, 1, \dots$$

Let  $F(u, v) = (v, f(v, u))$ . Then  $F$  maps  $I \times I$  into itself and is a cooperative map.

The second iterate  $T := F^2$  is given by

$$T(u, v) = (f(v, u), f(f(v, u), v))$$

and it is clearly strictly cooperative on  $\mathcal{I} \times \mathcal{I}$ .

If  $D_1g(u, v)$  and  $D_2g(u, v)$  denote the partial derivatives of a function  $g(u, v)$  with respect to  $u$  and  $v$ , the Jacobian matrix of  $T$  is

$$J_T(u, v) = \begin{pmatrix} D_2f(v, u) & D_1f(v, u) \\ D_1f(f(v, u), v) D_2f(v, u) & D_1f(f(v, u), v) D_1f(v, u) + D_2f(f(v, u), v) \end{pmatrix}. \quad (8)$$

The determinant of (8) is given by

$$\det J_T(u, v) = D_2f(v, u) D_2f(f(v, u), v) > 0.$$

To check injectivity of  $T$  I set

$$T((u_1, v_1)) = T((u_2, v_2))$$

which implies

$$f(v_1, u_1) = f(v_2, u_2), \quad f(f(v_1, u_1), v_1) = f(f(v_2, u_2), v_2)$$

and so  $f(f(v_1, u_1), v_1) = f(f(v_1, u_1), v_2)$ . By using the monotonicity of  $f$  I conclude that  $v_1 = v_2$  which in view of  $f(v_1, u_1) = f(v_2, u_2)$  gives  $u_1 = u_2$ .

The theory of monotone maps has been extensively developed at the level of ordered Banach spaces and applied to many types of equations such as ordinary, partial and discrete, see [22, 5, 42, 43, 82, 23, 24]. In particular, [42] has an extensive updated bibliography of different aspects of the theory of monotone maps. The theory of monotone discrete maps is more specialized and so one should expect stronger results in this case. An excellent review of basic results is given in [42, 43].

In particular, two-dimensional discrete maps are studied in great details and very precise results which describe the global dynamics and the basins of attractions of equilibrium points and period-two solutions as well as global stable manifolds are given in [22, 2, 65, 66, 13, 82].

### 1.3 Main Results

Let  $I$  be some interval of real numbers and let  $f \in C^1[I \times I, I]$  be strongly increasing function. Assume that for  $(x_0, y_0) \in I \times I$  there exists  $n_0$  such that  $F^n(x_0, y_0) \in [U_1, U_2]^2$  for all  $n > n_0$  where  $[U_1, U_2] \subseteq I$  and  $-\infty < U_1 < U_2 < \infty$  and assume that  $[U_1, U_2]^2$  is an invariant set for the map  $T$ , that is  $T : [U_1, U_2]^2 \rightarrow [U_1, U_2]^2$ . Let  $\bar{x}_0, \bar{x}_{SW}, \bar{x}_{NE} \in I$ ,  $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  be three equilibrium points of the difference equation Eq.(2) where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is unstable. Then the map  $F$  has three equilibrium solutions  $E_0(\bar{x}_0, \bar{x}_0)$ ,  $E_{SW}(\bar{x}_{SW}, \bar{x}_{SW})$  and  $E_{NE}(\bar{x}_{NE}, \bar{x}_{NE})$  such that  $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$  where the equilibrium points  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is unstable. By Theorem 3 all solutions converges to either equilibrium solutions or to the period-two solutions. As is well known [42, 43] the period-two solutions are the points in the South-East ordering, which means that they belong to  $Q_2(E_{SW}) \cup Q_4(E_{SW})$ . In the following discussion, I will assume that all period-two solutions belong to  $int(Q_2(E_{SW}) \cup Q_4(E_{SW}))$ .

Let  $\mathcal{B}(E_0)$  be the basin of attraction of  $E_0$  and  $\mathcal{B}(E_{NE})$  be the basin of attraction of  $E_{NE}$  with respect to the map  $T$ .

The following lemma holds.

**Lemma 1** *Let  $Q_1((x_0, y_0)) = \{(x, y) : x \geq x_0 \text{ and } y \geq y_0\} \cap I \times I$  and  $Q_3((x_0, y_0)) = \{(x, y) : x \leq x_0 \text{ and } y \leq y_0\} \cap I \times I$ . Then the following holds:*

- (i) *If there are no minimal period-two solutions in  $int(Q_3(E_{SW}))$  then  $int(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$ .*

(ii) If there are no minimal period-two solutions in  $\text{int}(Q_1(E_{SW}))$  then  $\text{int}(Q_1(E_{SW})) \subset \mathcal{B}(E_{NE})$ .

**Proof.** First, in view of Corollary 4  $\text{int}[(U_1, U_1), E_0] \subset \mathcal{B}(E_0)$  and also  $\text{int}[E_{NE}, (U_2, U_2)] \subset \mathcal{B}(E_{NE})$ . By Corollary 4 I obtain  $\text{int}(Q_3(E_{SW}) \cap Q_1(E_0)) \subset \mathcal{B}(E_0)$  and  $\text{int}(Q_1(E_{SW}) \cap Q_3(E_{NE})) \subset \mathcal{B}(E_{NE})$ . Since  $(U_1, U_1) \preceq_{ne} T(U_1, U_1) \preceq_{ne} E_0$  and  $T$  is cooperative map I obtain  $T^n(U_1, U_1) \rightarrow E_0$  as  $n \rightarrow \infty$ . For  $(x_0, y_0) \in \text{int}(Q_3(E_0))$  I have that  $(U_1, U_1) \preceq_{ne} (x_{-1}, x_0) \preceq_{ne} E_0$  which implies  $T^n(x_{-1}, x_0) \rightarrow E_0$  as  $n \rightarrow \infty$  i.e.  $\text{int}(Q_3(E_0)) \subset \mathcal{B}(E_0)$ . Assume that  $(x_0, y_0) \in \text{int}(Q_3(E_{SW}))$ . Then, there exists  $(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_3(E_0))$  such that  $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0)$  and  $(\tilde{x}_1, \tilde{y}_1) \in \text{int}(Q_3(E_{SW}) \cap Q_1(E_0))$  such that  $(x_0, y_0) \preceq_{ne} (\tilde{x}_1, \tilde{y}_1)$ . By monotonicity of  $T$  I have  $T^n(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(\tilde{x}_1, \tilde{y}_1)$  which implies  $T^n(x_0, y_0) \rightarrow E_0$  as  $n \rightarrow \infty$ . This implies that  $\text{int}(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$ . The proof of (ii) is similar and I skip it.  $\square$

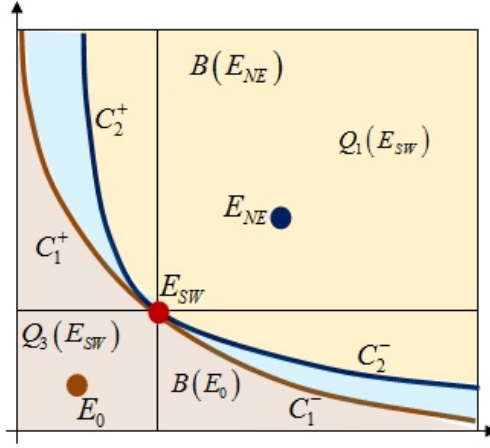


Figure 1. Visual illustration of Lemma 21.

Let  $\mathcal{C}_1^+$  denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_2(E_{SW})$  (the second quadrant relative to  $(E_{SW})$  and  $\mathcal{C}_1^-$  denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_4(E_{SW})$  (the fourth quadrant relative to  $(E_{SW})$ ). Also, let  $\mathcal{C}_2^+$  denote the boundary of  $\mathcal{B}(E_{NE})$  considered as a subset of  $Q_2(E_{SW})$  and  $\mathcal{C}_2^-$

denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_4(E_{SW})$ . It is easy to see that  $E_{SW} \in \mathcal{C}_1^+$ ,  $E_{SW} \in \mathcal{C}_1^-$ ,  $E_{SW} \in \mathcal{C}_2^+$ ,  $E_{SW} \in \mathcal{C}_2^-$ .

The proof of the following Lemma for cooperative map is the same as the proof of Claims 1 and 2 [4] for competitive map, so I skip it (See Figure 1).

**Lemma 2** *Let  $\mathcal{C}_1^+$  and  $\mathcal{C}_1^-$  be the sets defined above. Then the sets  $\mathcal{C}_1^+ \cup \mathcal{C}_1^-$  and  $\mathcal{C}_2^+ \cup \mathcal{C}_2^-$  are invariant under the map  $T$  and they are the graphs of continuous strictly decreasing functions.*

By Lemma 21 it remains to determine the behavior of the orbits of initial conditions  $(x_0, y_0)$  such that  $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0) \preceq (\bar{x}_0, \bar{y}_0)$  for some  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$  and  $(\bar{x}_0, \bar{y}_0) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .

Now, I present global dynamics of Eq.(2) depending on existence or non-existence of period-two solutions.

**Theorem 5** *Assume that Eq.(2) has no minimal period-two solutions and has three equilibrium points  $\bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is saddle point. Then there exist two continuous curves  $\mathcal{W}^s(E_{SW})$  and  $\mathcal{W}^u(E_{SW})$ , both passing through the point  $E_{SW}$ , such that  $\mathcal{W}^s(E_{SW})$  is a graph of decreasing function and  $\mathcal{W}^u(E_{SW})$  is a graph of an increasing function. The set of initial conditions  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq U_1, x_0 \geq U_1\} \cap I$  is the union of three disjoint basins of attraction, namely  $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$ , where  $\mathcal{B}(E_{SW}) = \mathcal{W}^s(E_{SW})$  and*

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_{-1}, x_0) | (x_{-1}, x_0) \preceq_{ne} (x_{E_0}, y_{E_0}) \\ &\quad \text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\} \\ \mathcal{B}(E_{NE}) &= \{(x_{-1}, x_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_{-1}, x_0) \\ &\quad \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}. \end{aligned}$$

Thus, I have  $\mathcal{W}^s(E_{SW}) = \mathcal{C}_1^+ \cup \mathcal{C}_1^- = \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .



**Proof.** By assumption the map  $T$  has three equilibrium point namely  $E_0$ ,  $E_{SW}$  and  $E_{NE}$  such that  $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$ . In this case,  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is a saddle point. Since  $f$  is strongly increasing then the map  $T$  is strongly cooperative on  $[U_1, \infty)^2$ . It follows from the Perron-Frobenius Theorem and a change of variables [82] that at each point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly cooperative then no eigenvector is aligned with a coordinate axis.

Hence, all conditions of Theorems 1 and 4 from [66] for cooperative map  $T$  are satisfied, which yields the existence of the global stable manifold  $\mathcal{W}^s(E_{SW})$  and the global unstable manifold  $\mathcal{W}^u(E_{SW})$ , where  $\mathcal{W}^s(E_{SW})$  is passing through the point  $E_{SW}$ , and it is a graph of decreasing function. The curve  $\mathcal{W}^u(E_{SW})$  is the graph of an increasing function. Let

$$\mathcal{W}^- = \{(x_{-1}, x_0) | (x_{-1}, x_0) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\},$$

$$\mathcal{W}^+ = \{(x_{-1}, x_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_{-1}, x_0) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}.$$

Take  $(x_0, y_0) \in \mathcal{W}^- \cap [U_1, \infty)^2$  and  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [U_1, \infty)^2$ . By Theorem 4 [66] I have that there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(E_{SW}))$  and  $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E_{SW}))$  for  $n > n_0$ . The rest of the proof follows from Lemma 20. See Figure 2 a) for its visual illustration.  $\square$

**Theorem 6** *Assume that Eq.(2) has no minimal period-two solutions and there exists three equilibrium points  $\bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is non-hyperbolic. Then, there exist two continuous curves  $\mathcal{C}_1(E_{SW})$  and  $\mathcal{C}_2(E_{SW})$  passing through the point  $E_{SW}$ , which are the graphs of decreasing functions. The set of initial condition  $Q_1 = \{(x_{-1}, x_0)\}$*

is the union of three disjoint basins of attraction, namely

$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$ , where

$$\mathcal{B}(E_0) = \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}_1(E_{SW})\}$$

$$\mathcal{B}(E_{SW}) = \{(x_0, y_0) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}_1(E_{SW}) \text{ and } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_2(E_{SW})\}$$

$$\mathcal{B}(E_{NE}) = \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_2(E_{SW})\}.$$

**Proof.** The characterization of  $\mathcal{B}(E_0)$  and  $\mathcal{B}(E_{NE})$  follows as in Theorem 26.

The existence of the curves  $\mathcal{C}_1(E_{SW})$  and  $\mathcal{C}_2(E_{SW})$  passing through the point  $E_{SW}$  which are the graphs of decreasing functions follow from Lemma 21. Assume that  $(x_{E_0}, y_{E_0}) \in \mathcal{C}_1(E_{SW})$ . Since  $\mathcal{C}_1(E_{SW})$  is invariant set and there are no period-two solutions it must be that  $T^n(x_{E_0}, y_{E_0}) \rightarrow E_{SW}$  as  $n \rightarrow \infty$ . Similarly, I obtain  $T^n(x_{E_{NE}}, y_{E_{NE}}) \rightarrow E_{SW}$  as  $n \rightarrow \infty$  if  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_2(E_{SW})$ . Suppose that  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_0}, y_{E_0}) \in \mathcal{C}_1(E_{SW})$  and  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_2(E_{SW})$ . Then  $T^n(x_{E_0}, y_{E_0}) \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(x_{E_{NE}}, y_{E_{NE}})$  which implies that  $T^n(x_0, y_0) \rightarrow E_{SW}$  as  $n \rightarrow \infty$ . See Figure 2 b) for the visual illustration of this result.  $\square$

Now, I consider the dynamical scenarios when there exists a minimal period-two solution which is a saddle point (see Figure 3 a)).

**Theorem 7** Assume that Eq.(2) has three equilibrium points  $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable. Further, assume that there exists a minimal period-two solution  $\{\Phi_1, \Psi_1\}$  which is a saddle point such that  $(\Phi_1, \Psi_1) \in \text{int}(Q_2(E_{SW}))$ . In this case there exist four continuous curves  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$ , where  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. The curves  $\mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$  are the graphs of increasing functions and are starting at  $E_0$ . Every solution which starts below

$\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$  in the North-east ordering converges to  $E_0$  and every solution which starts above  $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$  in the North-east ordering converges to  $E_{NE}$ , i.e.  $\mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$  and  $\mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_1^- = \mathcal{C}_2^-$ .

**Proof.** The map  $T$  is cooperative and all conditions of Theorems 1. and 4. from [66] are satisfied, which yields the existence of the global stable manifolds  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$  and the global unstable manifolds  $\mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$  where  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$  are passing through the point  $E_{SW}$ , and they are graphs of decreasing functions. Let

$$\begin{aligned}\mathcal{W}^- &= \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0}) \\ &\quad \text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\}, \\ \mathcal{W}^+ &= \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0) \\ &\quad \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\}.\end{aligned}$$

Take  $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$  and  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$ . By Theorem 4. [66] I have that there exist  $n_0, n_1 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_3(\Psi_1, \Phi_1))$  for  $n > n_0$  and  $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(\Phi_1, \Psi_1) \cap Q_1(\Psi_1, \Phi_1))$  for  $n > n_1$ . Since  $\text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_3(\Psi_1, \Phi_1)) \subseteq \text{int}(Q_3(E_{SW})) \subseteq \mathcal{B}(E_0)$  and  $\text{int}(Q_1(\Phi_1, \Psi_1) \cap Q_1(\Psi_1, \Phi_1)) \subseteq \text{int}(Q_1(E_{SW})) \subseteq \mathcal{B}(E_{NE})$  I have that  $T^n(x_0, y_0) \rightarrow E_0$  as  $n \rightarrow \infty$  if  $(x_0, y_0) \in \text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_3(\Psi_1, \Phi_1))$  and  $T^n(x_0, y_0) \rightarrow E_{NE}$  as  $n \rightarrow \infty$  if  $(x_0, y_0) \in \text{int}(Q_1(\Phi_1, \Psi_1) \cap Q_1(\Psi_1, \Phi_1))$ .  $\square$

Now, I consider the dynamical scenario when there exist three minimal period-two solutions.

**Theorem 8** *Assume that Eq.(2) has three equilibrium points  $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is a repeller. Further, assume that there exist three minimal period-two solutions  $\{\Phi_i, \Psi_i\}$ , such that  $(\Phi_i, \Psi_i) \in \text{int}(Q_2(E_{SW}))$ ,  $i = 1, 2, 3$ , where*

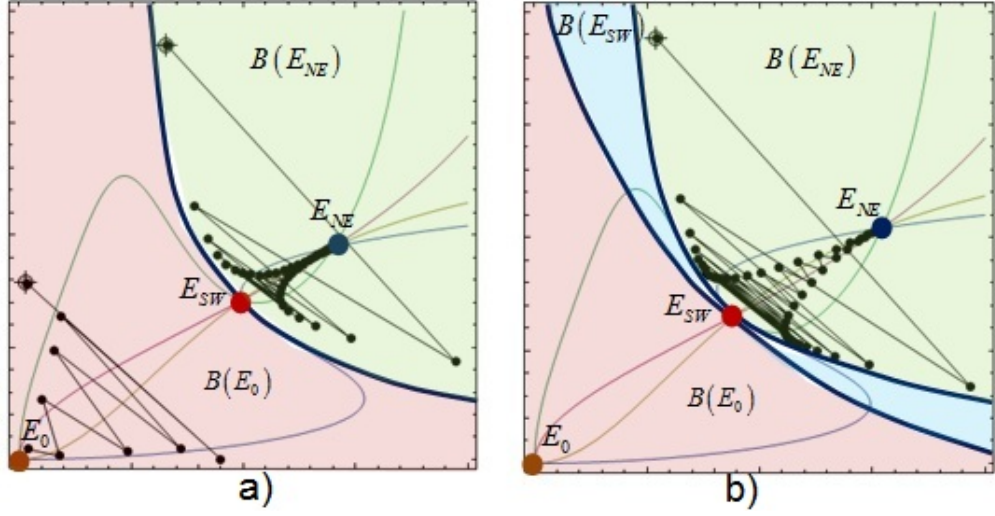


Figure 2. a) Visual illustration of Theorems 26 when  $a = 0.06$ ,  $b = 1.5$ . The case when  $E_{SW}$  is a saddle point. b) Visual illustration of 1 when  $a = 0.3$ ,  $b = 2$ . The case when  $E_{SW}$  is a saddle point.

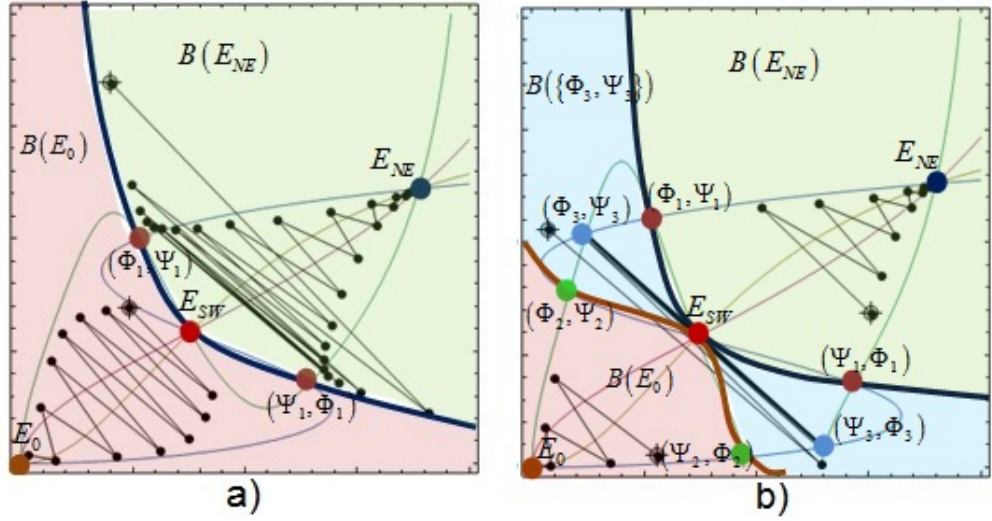


Figure 3. a) Visual illustration of Theorem 27 when  $a = 0.1$ ,  $b = 1.5$ . The case when  $E_{SW}$  is a repeller and  $\{\Phi_1, \Psi_1\}$  is a period two-solution which is a saddle point. b) Visual illustration of Theorem 28 when  $a = 0.1$ ,  $b = 2$ . The case when  $E_{SW}$  is a repeller,  $\{\Phi_1, \Psi_1\}$  and  $\{\Phi_2, \Psi_2\}$  are period two-solutions which are saddle points and  $\{\Phi_3, \Psi_3\}$  is the period two-solution which is locally asymptotically stable.

$\{\Phi_1, \Psi_1\}$  and  $\{\Phi_2, \Psi_2\}$  are the saddle points and  $\{\Phi_3, \Psi_3\}$  is locally asymptotically stable. In this case there exist four continuous curves  $\mathcal{W}^s(\Phi_1, \Psi_1)$ ,  $\mathcal{W}^s(\Psi_1, \Phi_1)$ ,  $\mathcal{W}^s(\Phi_2, \Psi_2)$ ,  $\mathcal{W}^s(\Psi_2, \Phi_2)$  where  $\mathcal{W}^s(\Phi_1, \Psi_1)$ ,  $\mathcal{W}^s(\Psi_1, \Phi_1)$ ,  $\mathcal{W}^s(\Phi_2, \Psi_2)$ ,  $\mathcal{W}^s(\Psi_2, \Phi_2)$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. Every solution which starts below  $\mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2)$  in the North-east ordering converges to  $E_0$  and every solution which starts above  $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$  in the North-east ordering converges to  $E_{NE}$ . Every solution which starts above  $\mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2)$  and below  $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$  in the North-east ordering converges to  $\{\Phi_3, \Psi_3\}$ . In other words, the set of initial condition  $Q_1 = \{(x_{-1}, x_0)\}$  is the union of five disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{\Phi_1, \Psi_1\}) \cup \mathcal{B}(\{\Phi_2, \Psi_2\}) \cup \mathcal{B}(\{\Phi_3, \Psi_3\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{\Phi_1, \Psi_1\}) = \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1),$$

$$\mathcal{B}(\{\Phi_2, \Psi_2\}) = \mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2),$$

$$\mathcal{B}(E_0) = \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2)\},$$

$$\mathcal{B}(E_{NE}) = \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\},$$

$$\mathcal{B}(\{\Phi_3, \Psi_3\}) = \{(x_0, y_0) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1) \text{ and}$$

$$(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2)\}.$$

Thus, I have  $\mathcal{W}^s(\Phi_2, \Psi_2) = \mathcal{C}_1^+$ ,  $\mathcal{W}^s(\Psi_2, \Phi_2) = \mathcal{C}_1^-$ ,  $\mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_2^+$ , and  $\mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_2^-$ .

**Proof.** All conditions of Theorems 1. and 4. in [66] for cooperative map  $T$  are satisfied, which yields the existence of the global stable manifolds

$\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{W}^s(\Phi_2, \Psi_2), \mathcal{W}^s(\Psi_2, \Phi_2)$  which are passing through the point  $E_{SW}$ , and they are graphs of decreasing functions.

Since  $T$  is cooperative map it can be seen that  $(\Phi_1, \Psi_1) \ll_{ne} (\Phi_3, \Psi_3) \ll_{ne} (\Phi_2, \Psi_2)$  or  $(\Phi_2, \Psi_2) \ll_{ne} (\Phi_3, \Psi_3) \ll_{ne} (\Phi_1, \Psi_1)$ .

Assume  $(\Phi_1, \Psi_1) \ll_{ne} (\Phi_3, \Psi_3) \ll_{ne} (\Phi_2, \Psi_2)$ . As in the proof of Theorem 27 one can see that

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\}, \\ \mathcal{B}(E_{NE}) &= \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0) \\ &\quad \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_2, \Psi_2) \cup \mathcal{W}^s(\Psi_2, \Phi_2)\}. \end{aligned}$$

So, I assume that  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1)$ .

By Theorem 4. in [66] I have that there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in \text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_1(\Phi_2, \Psi_2))$  for  $n > n_0$ . By Corollary 4 I get  $[(\Phi_2, \Psi_2), (\Phi_3, \Psi_3)] \cup [(\Phi_3, \Psi_3), (\Phi_1, \Psi_1)] \subseteq \mathcal{B}(\Phi_3, \Psi_3)$  which implies that  $\text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_1(\Phi_2, \Psi_2)) = [(\Phi_2, \Psi_2), (\Phi_1, \Psi_1)] \subseteq \mathcal{B}(\Phi_3, \Psi_3)$ . If  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_2, \Psi_2)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1)$  then there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in \text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_1(\Phi_2, \Psi_2))$  for  $n > n_0$ . By Corollary 4 I get  $[(\Psi_2, \Phi_2), (\Psi_3, \Phi_3)] \cup [(\Psi_3, \Phi_3), (\Psi_1, \Phi_1)] \subseteq \mathcal{B}(\Phi_3, \Psi_3)$  which implies that  $\text{int}(Q_3(\Psi_1, \Phi_1) \cap Q_1(\Psi_2, \Phi_2)) = [(\Psi_2, \Phi_2), (\Psi_1, \Phi_1)] \subseteq \mathcal{B}(\Phi_3, \Psi_3)$ . This completes the proof.  $\square$

Now, I consider two dynamical scenarios when there exists a minimal period-two solution  $\{\Phi, \Psi\}$  which is a non-hyperbolic of stable type (i.e. if  $\mu_1$  and  $\mu_2$  are eigenvalues of  $J_T(\Phi, \Psi)$  then  $\mu_1 = 1$  and  $|\mu_2| < 1$ ).

**Theorem 9** *Assume that Eq.(2) has three equilibrium points  $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and*

$\bar{x}_{SW}$  is a repeller or non-hyperbolic equilibrium point. Further, assume that there exist two minimal period-two solutions  $\{\Phi, \Psi\}$  and  $\{\Phi_1, \Psi_1\}$ , where  $\{\Phi, \Psi\}$  is a non-hyperbolic period-two solution of the stable type and  $\{\Phi_1, \Psi_1\}$  is a saddle point, and  $(\Phi, \Psi) \ll_{ne} (\Phi_1, \Psi_1)$  (See Figure 4 a). In this case there exist four continuous curves  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{C}^s(\Phi, \Psi), \mathcal{C}^s(\Psi, \Phi)$  which are passing through the point  $E_{SW}$  and which are graphs of decreasing functions. The set  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq U_1, x_0 \geq U_1\}$  is the union of four disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{\Phi_1, \Psi_1\}) \cup \mathcal{B}(\{\Phi, \Psi\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{\Phi_1, \Psi_1\}) = \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1),$$

$$\mathcal{B}(E_0) = \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)\},$$

$$\mathcal{B}(E_{NE}) = \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\},$$

$$\mathcal{B}(\{\Phi, \Psi\}) = \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi) \cup \{(x_0, y_0) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)\}.$$

Thus, I have

$$\mathcal{C}(\Phi, \Psi) = \mathcal{C}_1^+, \mathcal{C}(\Psi, \Phi) = \mathcal{C}_1^-, \mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_2^+, \text{ and } \mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_2^-.$$

**Proof.** Since  $\{\Phi, \Psi\}$  is a non-hyperbolic period-two solution of the stable type and  $\{\Phi_1, \Psi_1\}$  is a saddle point, all conditions of Theorems 1 and 4 [66] for cooperative map  $T$  are satisfied, which yields the existence of the global stable manifolds  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$  and invariant curves  $\mathcal{C}(\Phi, \Psi), \mathcal{C}(\Psi, \Phi)$  which are passing through the point  $E_{SW}$ , and they are graphs of decreasing functions.

Take  $(x_0, y_0)$  such that  $(x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0})$  for some  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)$ . In view of Theorem 4 [66] there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(\Phi, \Psi) \cap Q_3(\Psi, \Phi))$  for  $n > n_0$ . Since  $\text{int}(Q_3(\Phi, \Psi) \cap Q_3(\Psi, \Phi)) \subseteq \text{int}(Q_3(E_{SW}))$  by Lemma 20 I obtain

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)\}.$$

If  $(x_0, y_0)$  is such that  $(x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0)$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$  by Theorem 4 [66] there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_1(\Phi_1, \Psi_1) \cap Q_1(\Psi_1, \Phi_1))$  for  $n > n_0$ . Since  $\text{int}(Q_1(\Phi_1, \Psi_1) \cap Q_1(\Psi_1, \Phi_1)) \subseteq \text{int}(Q_1(E_{SW}))$  by Lemma 20 I obtain

$$\mathcal{B}(E_{NE}) = \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\}.$$

Now, I assume that  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Phi_1, \Psi_1)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(\Phi, \Psi)$ . By Theorem 4 [66] I have that there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(\Phi_1, \Psi_1) \cap Q_1(\Phi, \Psi))$  for  $n > n_0$ . By Corollary 4 I get  $[[\Phi, \Psi], (\Phi_1, \Psi_1)] \subseteq \mathcal{B}(\{\Phi, \Psi\})$ . Similarly, if  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(\Psi_1, \Phi_1)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(\Psi, \Phi)$  I have that there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(\Psi_1, \Phi_1) \cap Q_1(\Psi, \Phi))$  for  $n > n_0$ . By Corollary 4 I get  $[[T(P), (\Psi_1, \Phi_1)]] \subseteq \mathcal{B}(\{\Phi, \Psi\})$ . This implies  $(x_0, y_0) \in \mathcal{B}(\{\Phi, \Psi\})$ .  $\square$

The proof of the following result is similar to the proof of Theorem 29 and will be omitted.

**Theorem 10** *Assume that Eq.(2) has three equilibrium points  $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is a repeller or non-hyperbolic equilibrium point. Further, assume that there exist two minimal period-two solutions  $\{\Phi, \Psi\}$  and  $\{\Phi_1, \Psi_1\}$ , where  $\{\Phi, \Psi\}$  is a*



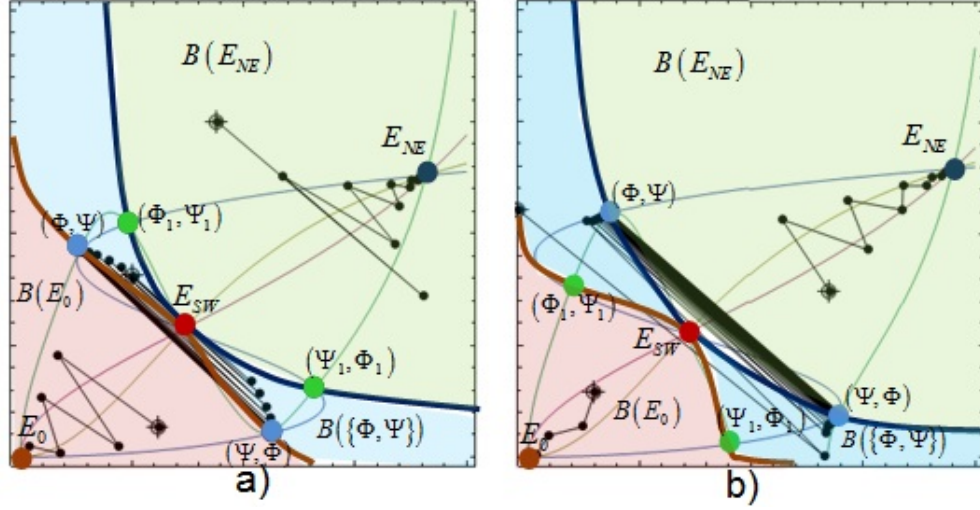


Figure 4. a) Visual illustration of Theorem 29 when  $a = 0.06$ ,  $b = 1.9998768381155188$ . The case when  $E_{SW}$  is a repeller,  $\{P, T(P)\}$  is the period two-solution which is non-hyperbolic and  $\{\Phi_1, \Psi_1\}$  is the period two-solution which is a saddle point. b) Visual illustration of the Theorem 30 when  $a = 0.1$ ,  $b = 1.97282$ . The case when  $E_{SW}$  is a repeller,  $\{\Phi, \Psi\}$  is the period two-solution which is non-hyperbolic and  $\{\Phi_2, \Psi_2\}$  is the period two-solution which is a saddle point.

*non-hyperbolic period-two solution of the stable type and  $\{\Phi_1, \Psi_1\}$  is a saddle point, and  $(\Phi_1, \Psi_1) \ll_{ne} (\Phi, \Psi)$  (See Figure 4 b). In this case there exist four continuous curves  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{C}^s(\Phi, \Psi), \mathcal{C}^s(\Psi, \Phi)$  where  $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{C}(\Phi, \Psi), \mathcal{C}(\Psi, \Phi)$  are passing through the point  $E_{SW}$ , which are graphs of decreasing functions. The set  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq U_1, x_0 \geq U_1\}$  is the union of four disjoint basins of attraction, namely*

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{\Phi_1, \Psi_1\}) \cup \mathcal{B}(\{\Phi, \Psi\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{\Phi_1, \Psi_1\}) = \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1),$$

$$\mathcal{B}(E_0) = \{(x_0, y_0) | (x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\},$$

$$\mathcal{B}(E_{NE}) = \{(x_0, y_0) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)\},$$

$$\mathcal{B}(\{\Phi, \Psi\}) = \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi) \cup \{(x_0, y_0) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(\Phi, \Psi) \cup \mathcal{C}(\Psi, \Phi)$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)\}.$$

Thus, I have

$$\mathcal{C}(\Phi, \Psi) = \mathcal{C}_2^+, \mathcal{C}(\Psi, \Phi) = \mathcal{C}_2^-, \mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_1^+, \text{ and } \mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_1^-.$$

## 1.4 Examples

In this section I give an application of our results in establishing global dynamics of some equations from mathematical biology. I present two cases in detail. All three models are of the types

$$x_{n+1} = f_1(x_n) + f_2(x_{n-1}), \quad n = 0, 1, \dots,$$

where  $f_i, i = 1, 2$  are transition functions, considered by many researchers in mathematical biology [74]. The most common transition functions in modeling are linear, Beverton-Holt ( $f(u) = \frac{au}{1+u}$ ) and sigmoid Beverton-Holt ( $f(u) = \frac{au^\sigma}{1+u^\sigma}, \sigma \geq 1$ ). The case when both transition functions are Beverton-Holt functions was treated in [74] and I prove that in this case there are no period-two solutions and so every solution converges to an equilibrium. However, in all other cases for any other combination of transition functions the period-two solutions exist and play an important role in the dynamics.

### 1.4.1 Example 1: $x_{n+1} = \frac{Ax_n^2}{1+x_n^2} + \frac{Bx_{n-1}^2}{1+x_{n-1}^2}$

In this part I apply Theorems 26-30 to describe the global dynamics of difference equation in the title.

#### The equilibrium points

I consider the difference equation

$$x_{n+1} = \frac{Ax_n^2}{1+x_n^2} + \frac{Bx_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \dots \quad (9)$$

where  $A, B > 0$  and the initial conditions  $x_{-1}, x_0$  are non-negative. In view of the above restriction on the initial conditions of Eq.(9), the equilibrium points of Eq.(9) are the positive solutions of the equation

$$\bar{x} = \frac{A\bar{x}^2}{1+\bar{x}^2} + \frac{B\bar{x}^2}{1+\bar{x}^2}$$

or equivalently

$$\bar{x}^3 - \bar{x}^2(A+B) + \bar{x} = 0. \quad (10)$$

Now I have the following result.

**Lemma 3** *The following holds:*

- i) If  $A+B < 2$  then equation (9) has a unique equilibrium point  $x_0 = 0$ ;
- ii) If  $A+B = 2$  then equation (9) has two equilibrium points  $x_0 = 0$  and  $x = (A+B)/2$ ;
- iii) If  $A+B > 2$  then equation (9) has three equilibrium points  $x_0 = 0$ ,  $x_{SW} = \frac{1}{2} \left( A+B - \sqrt{(A+B)^2 - 4} \right)$  and  $x_{NE} = \frac{1}{2} \left( A+B + \sqrt{(A+B)^2 - 4} \right)$  and  $x = (A+B)/2$ .

Next, I investigate the stability of the positive equilibrium points of Eq.(9). Set

$$f(u, v) = \frac{Au^2}{1+u^2} + \frac{Bv^2}{1+v^2}$$

and observe that  $f_u(u, v) > 0$  and  $f_v(u, v) > 0$ . The next three lemmas are straightforward.

**Lemma 4** *Assume that  $A, B > 0$ . Then the equilibrium point  $x_0$  is locally asymptotically stable.*

**Lemma 5** *Assume that  $A + B \geq 2$ . Then the equilibrium point  $x_{NE}$  is locally asymptotically stable if  $A + B > 2$  and non-hyperbolic if  $A + B = 2$ .*

**Proof.** It can be seen that

$$p = f_u(x_{NE}, x_{NE}) = \frac{4A}{(A+B)^2 \left( \sqrt{(A+B)^2 - 4} + A + B \right)}$$

and

$$q = f_v(x_{NE}, x_{NE}) = \frac{4B}{(A+B)^2 \left( \sqrt{(A+B)^2 - 4} + A + B \right)}.$$

Then, the proof follows from Theorem 1.1.1 in [54] and the fact that

$$1 - p - q = \frac{\sqrt{(A+B)^2 - 4}}{A+B} \text{ and } p, q > 0.$$

□

**Lemma 6** *Assume that  $A + B \geq 2$ . Then the equilibrium point  $x_{SW}$  is:*

- i) a saddle point if  $2A(A+B) + (A-B)\sqrt{(A+B)^2 - 4} > 0$  and  $A+B > 2$ ;*
- ii) a repeller if  $2A(A+B) + (A-B)\sqrt{(A+B)^2 - 4} < 0$  and  $A+B > 2$ ;*
- iii) a non-hyperbolic if  $A+B = 2$  or  $2A(A+B) + (A-B)\sqrt{(A+B)^2 - 4} = 0$ .*

**Proof.** It can be seen that

$$p = f_u(x_{SW}, x_{SW}) = \frac{A \left( \sqrt{(A+B)^2 - 4} + A + B \right)}{(A+B)^2},$$

$$q = f_v(x_{SW}, x_{SW}) = \frac{B \left( \sqrt{(A+B)^2 - 4} + A + B \right)}{(A+B)^2}.$$

Then, the proof follows from Theorem 1.1.1 in [54] and the fact that

$$1-p-q = -\frac{\sqrt{(A+B)^2-4}}{A+B} \text{ and } 1+p-q = \frac{2A(A+B) + (A-B)\sqrt{(A+B)^2-4}}{(A+B)^2}.$$

□

### Period-two solutions

Next, I investigate the existence and stability of the positive minimal period-two solutions of equation (9). Let  $\{\phi, \psi\}$  be a minimal period-two solution of equation (9). Then

$$\phi = f(\psi, \phi) \text{ and } \psi = f(\phi, \psi) \text{ with } \psi, \phi \in [0, \infty) \text{ and } \phi \neq \psi$$

which is equivalent to

$$\phi = \frac{A\psi^2}{1+\psi^2} + \frac{B\phi^2}{1+\phi^2} \text{ and } \psi = \frac{A\phi^2}{1+\phi^2} + \frac{B\psi^2}{1+\psi^2} \text{ with } \phi \neq \psi$$

which is true if and only if  $\phi \neq \psi$ ,

$$-A\psi^2 - A\psi^2\phi^2 - B\psi^2\phi^2 - B\phi^2 + \psi^2\phi^3 + \phi^3 + \psi^2\phi + \phi = 0 \quad (11)$$

and

$$\psi - A\psi^2\phi^2 - A\phi^2 - B\psi^2 - B\psi^2\phi^2 + \psi^3 + \psi + \psi^3\phi^2 + \psi\phi^2 = 0. \quad (12)$$

By eliminating  $\psi$  from (11) and (12) I obtain

$$\phi(\phi^2 + 1)(A\phi + B\phi - \phi^2 - 1)\tilde{f}(\phi) = 0$$

and by eliminating  $\phi$  from (11) and (12) I obtain

$$\psi(\psi^2 + 1)(A\psi + B\psi - \psi^2 - 1)\tilde{f}(\psi) = 0$$

where

$$\begin{aligned} \tilde{f}(x) = & B^2x^6 + 2Bx^5(A^2 - B^2) + x^4(A^4 - 2A^2B^2 + A^2 + B^4 + 2B^2) \\ & + x^3(A^3 + 2A^2B - AB^2 - 2B^3) + x^2(A^4 - A^3B - A^2(B^2 - 2) + AB^3 + B^2) \\ & + x(A^3 - AB^2) + A^2. \end{aligned}$$

Since  $(A + B)x - x^2 - 1 \neq 0$  for any  $x$  different from the equilibrium point, the minimal period-two solutions are solutions of the equation

$$\tilde{f}(x) = 0. \quad (13)$$

**Lemma 7** *Let*

$$\begin{aligned} \Delta &= 4A^6 - 12A^5B + 8A^4B^2 + 8A^4 + 8A^3B^3 - 36A^3B - 12A^2B^4 + 47A^2B^2 + 4A^2 + \\ &\quad 4AB^5 - 18AB^3 + 8AB - B^4 + 4B^2, \\ \Delta_1 &= 2A^4 - 4A^2B^2 - 3A^2 + 2B^4 - 6B^2, \\ \Delta_2 &= -(8A^7 - 8A^5B^2 + 16A^5 - 32A^4B - 8A^3B^4 - 45A^3B^2 + 8A^3 + 39A^2B^3) \\ &\quad 8A^2B + 8AB^6 - 40AB^4 + 16AB^2 - 4B^5 + 16B^3, \\ \Delta_3 &= 32A^{12} - 64A^{11}B - 96A^{10}B^2 + 148A^{10} + 256A^9B^3 - 436A^9B + 64A^8B^4 + \\ &\quad 332A^8B^2 + 264A^8 - 384A^7B^5 - 92A^7B^3 - 900A^7B + 64A^6B^6 + 324A^6B^4 + \\ &\quad 1627A^6B^2 + 212A^6 + 256A^5B^7 - 84A^5B^5 - 1992A^5B^3 - 624A^5B - 96A^4B^8 - \\ &\quad 700A^4B^6 + 1402A^4B^4 + 2008A^4B^2 + 64A^4 - 64A^3B^9 + 652A^3B^7 - 192A^3B^5 - \\ &\quad 1536A^3B^3 + 32A^2B^{10} - 104A^2B^8 - 425A^2B^6 + 724A^2B^4 - 128A^2B^2 - 40AB^9 + \\ &\quad 204AB^7 - 144AB^5 + 12B^8 - 64B^6 + 64B^4, \\ \Delta_4 &= \Delta (4A^9 - 16A^7B^2 + 15A^7 - 23A^6B + 24A^5B^4 + 10A^5B^2 + \\ &\quad 7A^5 - 20A^4B^3 - 77A^4B - 16A^3B^6 - A^3B^4 + 69A^3B^2 - 8A^3 + 45A^2B^5 - 77A^2B^3 + \\ &\quad 8A^2B + 4AB^8 - 24AB^6 + 20AB^4 + 8AB^2 - 2B^7 + 10B^5 - 8B^3), \\ \Delta_5 &= -3A^4 - 8A^3B - 6A^2B^2 - 4A^2 + 8AB + B^4 - 4B^2. \end{aligned}$$

*Then the following holds:*

- a) *Consider equation (13). Then, all its real roots are positive numbers. Furthermore, equation (9) has up to three minimal period-two solutions.*
- b) *If  $\Delta_i > 0$ , for all  $1 \leq i \leq 5$  and  $\Delta > 0$  then equation (13) has six real roots. Consequently, equation (9) has three minimal period-two solutions.*

c) If  $\Delta_i \leq 0$  for some  $1 \leq i \leq 4$  and  $\Delta_5 > 0, \Delta > 0$  then equation (13) has two distinct real roots and two pairs of conjugate imaginary roots. Consequently, equation (9) has one minimal period-two solution.

d) If  $\Delta_i > 0$  for all  $1 \leq i \leq j-1$  and  $\Delta_i < 0$  for all  $j \leq i \leq 4$  for some  $1 \leq j \leq 5$  and  $\Delta_5 < 0, \Delta > 0$  then equation (13) has four distinct real roots and one pair of conjugate imaginary roots. Consequently, Eq.(9) has two minimal period-two solutions.

e) If  $\Delta_i \leq 0, \Delta_{i+1} \geq 0$  for some  $1 \leq i \leq 3$  and  $\Delta_5 < 0, \Delta > 0$  then equation (13) has three pairs of conjugate imaginary roots. Consequently, Eq.(9) has no minimal period-two solution.

f) Assume that  $\Delta = 0$  and  $\Delta_3 \neq 0$  and  $\Delta_5 \neq 0$ .

f.1) If  $(\Delta_1 \leq 0 \text{ and } \Delta_2 \geq 0)$  or  $(\Delta_2 \leq 0 \text{ and } \Delta_3 > 0)$  then equation (13) has no real roots and has two distinct pairs of conjugate imaginary roots, one of them of multiplicity one and other one of multiplicity two. Consequently, equation (9) has no minimal period-two solutions.

f.2) If  $\Delta_1 > 0$  and  $\Delta_2 > 0$  and  $\Delta_3 > 0$  then equation (13) has four distinct real roots, two of them are multiplicity two and other two of multiplicity one and has no conjugate imaginary roots. Consequently, equation (9) has two minimal period-two solutions.

**Proof.** The proof of a) follows from Descartes' Rule of Signs.

The discrimination matrix [6, 15] of  $\tilde{f}(x)$  and  $\tilde{f}'(x)$  is given by

$$Discr(\tilde{f}) = \begin{pmatrix} a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{pmatrix}$$

where  $a_6 = B^2$ ,  $a_5 = 2B(A^2 - B^2)$ ,  $a_4 = A^4 - 2A^2B^2 + A^2 + B^4 + 2B^2$ ,  $a_3 = A^3 + 2A^2B - AB^2 - 2B^3$ ,  $a_2 = A^4 - A^3B - A^2(B^2 - 2) + AB^3 + B^2$ ,  $a_1 = A^3 - AB^2$  and  $a_0 = A^2$ .

Let  $D_k$  denote the determinant of the submatrix of  $Discr(\tilde{f})$ , formed by the first  $2k$  row and the first  $2k$  columns, for  $k = 1, \dots, m$ . By straightforward calculation one can see that

$$D_1 = 6B^4,$$

$$D_2 = 4B^6(2A^4 - 4A^2B^2 - 3A^2 + 2B^4 - 6B^2),$$

$$D_3 = -2B^6(A - B)^2(A + B)(8A^7 - 8A^5B^2 + 16A^5 - 32A^4B - 8A^3B^4 - 45A^3B^2 + 8A^3 + 39A^2B^3 + 8A^2B + 8AB^6 - 40AB^4 + 16AB^2 - 4B^5 + 16B^3),$$

$$D_4 = A^2B^6(B - A)^2(A + B)^2(32A^{12} - 64A^{11}B - 96A^{10}B^2 + 148A^{10} + 256A^9B^3 - 436A^9B + 64A^8B^4 + 332A^8B^2 + 264A^8 - 384A^7B^5 - 92A^7B^3 - 900A^7B + 64A^6B^6 + 324A^6B^4 + 1627A^6B^2 + 212A^6 + 256A^5B^7 - 84A^5B^5 - 1992A^5B^3 - 624A^5B - 96A^4B^8 - 700A^4B^6 + 1402A^4B^4 + 2008A^4B^2 + 64A^4 - 64A^3B^9 + 652A^3B^7 - 192A^3B^5 - 1536A^3B^3 + 32A^2B^{10} - 104A^2B^8 - 425A^2B^6 + 724A^2B^4 - 128A^2B^2 - 40AB^9 + 204AB^7 - 144AB^5 + 12B^8 - 64B^6 + 64B^4),$$



$$\begin{aligned}
D_5 = & 2A^4B^6(B-A)^4(A+B)^3(4A^6 - 12A^5B + 8A^4B^2 + 8A^4 + 8A^3B^3 - 36A^3B - \\
& 12A^2B^4 + 47A^2B^2 + 4A^2 + 4AB^5 - 18AB^3 + 8AB - B^4 + 4B^2) \\
& (4A^9 - 16A^7B^2 + 15A^7 - 23A^6B + 24A^5B^4 + 10A^5B^2 + 7A^5 - 20A^4B^3 - \\
& 77A^4B - 16A^3B^6 - A^3B^4 + 69A^3B^2 - 8A^3 + 45A^2B^5 - .77A^2B^3 + 8A^2B + 4AB^8 - \\
& 24AB^6 + 20AB^4 + 8AB^2 - 2B^7 + 10B^5 - 8B^3) \\
D_6 = & A^6B^6(B-A)^6(A+B)^4(-3A^4 - 8A^3B - 6A^2B^2 - 4A^2 + 8AB + B^4 - 4B^2) \\
& (4A^6 - 12A^5B + 8A^4B^2 + 8A^4 + 8A^3B^3 - 36A^3B - 12A^2B^4 + 47A^2B^2 + \\
& 4A^2 + 4AB^5 - 18AB^3 + 8AB - B^4 + 4B^2)^2.
\end{aligned}$$

The rest of the proof follows from Theorem 1 [15].  $\square$

### The global behavior

In this section I describe the global behavior of equation (9) which has three equilibrium points  $\bar{x}_0, \bar{x}_{SW}, \bar{x}_{NE} \in I$  such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is unstable. Further,  $x_n < A + B$  for all  $n \geq 1$ . One can see that all minimal period two solutions of (9) belong to  $\text{int}(Q_2(E_{SW}) \cup Q_4(E_{SW}))$ .

**Lemma 8** *If  $A + B < 2$  then there exists a unique equilibrium point  $x_0 = 0$  which is globally asymptotically stable.*

**Proof.** The proof follows from Theorem 2 and the fact that  $x_n < A + B$  for  $n \geq 1$ .

$\square$

**Theorem 11** *Assume that  $A + B > 2$ . Then the following holds:*

- i) If  $\Delta_i \leq 0$  and  $\Delta_{i+1} \geq 0$  for some  $1 \leq i \leq 3$  and  $\Delta_5 < 0, \Delta > 0$  then equation (9) has three equilibrium points such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is saddle point and has no period-two solution. The global behavior of equation (9) is described by Theorem 1. For example, this happens for  $A = 0.06$  and  $B = 1.5$ .*

ii) If  $\Delta_i \leq 0$  for some  $1 \leq i \leq 4$  and  $\Delta_5 > 0, \Delta > 0$  then equation (9) has three equilibrium points such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is repeller and one period-two solution  $\{\phi_1, \psi_1\}$  which is a saddle point. The global behavior of equation (9) is described by Theorem 27. For example, this happens for  $A = 0.06$  and  $B = 1.5$ .

iii) If  $\Delta_i > 0$ , for all  $1 \leq i \leq 5$  and  $\Delta > 0$  then equation (9) has three equilibrium points such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is repeller and three minimal period-two solutions  $\{\phi_1, \psi_1\}$ ,  $\{\phi_2, \psi_2\}$  and  $\{\phi_3, \psi_3\}$  such that  $(\phi_1, \psi_1) \preceq_{ne} (\phi_2, \psi_2) \preceq_{ne} (\phi_3, \psi_3)$  where  $\{\phi_1, \psi_1\}$ , and  $\{\phi_3, \psi_3\}$  are saddle points and  $\{\phi_2, \psi_2\}$  is locally asymptotically stable. The global behavior of equation (9) is described by Theorem 28. For example, this happens for  $A = 0.06$  and  $B = 2.09$ .

iv) If  $\Delta = 0, \Delta_5 > 0, \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$  then equation (9) has three equilibrium points such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is repeller and two minimal period-two solutions  $\{\phi_1, \psi_1\}$  and  $\{\phi_2, \psi_2\}$  such that  $(\phi_1, \psi_1) \preceq_{ne} (\phi_2, \psi_2)$  where either  $\{\phi_1, \psi_1\}$  is non-hyperbolic and  $\{\phi_2, \psi_2\}$  is a saddle point or  $\{\phi_1, \psi_1\}$  is a saddle point and  $\{\phi_2, \psi_2\}$  is non-hyperbolic. If period-two solution which is non-hyperbolic is of stable type then the global behavior of equation (9) is described by Theorems 28 and 29. For example, this happens for  $A = 0.06$  and  $B = 1.9998768381155188$ .

### **Proof.**

i) By assumption I have that  $\Delta_5 < 0$ . By Lemmas 4 and 5 the equilibrium

points  $x_0$  and  $x_{NE}$  are locally asymptotically stable. Since

$$(2A^2 + 2AB)^2 - (A - B)^2 ((A + B)^2 - 4) = 3A^4 + 8A^3B + 6A^2B^2 + 4A^2 - 8AB - B^4 + 4B^2 \quad (14)$$

by Lemma 6, the equilibrium point  $x_{SW}$  is a saddle point. By Lemma 17 there are no minimal period-two solutions so the rest of the proof follows from Theorem 1.

ii) By our assumptions I have that  $\Delta_5 > 0$ . By Lemmas 4 and 5 the equilibrium points  $x_0$  and  $x_{NE}$  are locally asymptotically stable. In view of (14) and Lemma 6 the equilibrium point  $x_{SW}$  is a repeller. Since the discriminant of polynomial  $\tilde{f}(x)$  is  $Dis(\tilde{f}) = D_6 \neq 0$  similarly as in Theorem 15 [4] one can see that all period-two solutions are hyperbolic. In view of Lemma 21 I have that  $\mathcal{C}_1^+ \cup \mathcal{C}_1^-$  is a totally ordered set which is invariant under  $T$ . If  $(x_0, y_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$  then  $\{T^n(x_0, y_0)\}$  is eventually componentwise monotone. Then there exists minimal period-two solution  $\{(\phi_1, \psi_1), (\psi_1, \phi_1)\} \in \mathcal{C}_1^+ \cup \mathcal{C}_1^- \subset Q_2(E_{SW}) \cup Q_4(E_{SW})$  such that  $T^n(x_0, y_0) \rightarrow (\phi_1, \psi_1)$  as  $n \rightarrow \infty$ . By Lemma 17 there exists only one minimal period-two solution, which implies that  $\{\phi_1, \psi_1\}$  is a saddle point. The rest of the proof follows from Theorem 27.

iii) Similarly as in ii) one can see that  $(\phi_1, \psi_1) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$  and  $(\phi_3, \psi_3) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$  which are saddle points. By Corollary 4 there exists minimal period-two solution  $\{\phi_2, \psi_2\}$  such that  $(\phi_1, \psi_1) \preceq_{ne} (\phi_2, \psi_2) \preceq_{ne} (\phi_3, \psi_3)$  which is locally asymptotically stable. The rest of the proof follows from Theorem 28.

iv) Since  $\Delta_5 > 0$ , by Lemmas 4 and 5 the equilibrium points  $x_0$  and  $x_{NE}$  are locally asymptotically stable. Since

$$(2A^2 + 2AB)^2 - (A - B)^2 ((A + B)^2 - 4) = 3A^4 + 8A^3B + 6A^2B^2 + 4A^2 - 8AB - B^4 + 4B^2$$

by Lemma 6 the equilibrium point  $x_{SW}$  is a repeller. By Lemma 17 there exists two minimal period-two solutions. Let  $g(x, y) = f(y, x)$  and  $h(x, y) = f(f(y, x), y)$ . Then, period-two curves, that is the curves which intersection is a period-two solution, are given by

$$\mathcal{C}_{\bar{F}} := \{(x, y) : g(x, y) = x\}, \quad \mathcal{C}_{\bar{G}} := \{(x, y) : h(x, y) = y\}.$$

Taking derivatives of  $g(x, y) = x$  and  $h(x, y) = y$  with respect to  $x$  I get

$$y'_{\bar{F}}(x) = \frac{1 - g'_x(x, y)}{g'_y(x, y)}, \quad y'_{\bar{G}}(x) = \frac{h'_x(x, y)}{1 - h'_y(x, y)}.$$

One can see that

$$\begin{aligned} y'_{\bar{F}}(\phi) - y'_{\bar{G}}(\phi) &= \frac{1 - g'_x(\phi, \psi)}{g'_y(\phi, \psi)} - \frac{h'_x(\phi, \psi)}{1 - h'_y(\phi, \psi)} \\ &= \frac{1 - e_1}{f_1} - \frac{g_1}{1 - h_1} = \frac{1 - (e_1 + h_1) + (e_1 h_1 - f_1 g_1)}{f_1(1 - h_1)} \\ &= \frac{p(1)}{f_1(1 - h_1)} = \frac{(1 - \mu_1)(1 - \mu_2)}{f(1 - h)}, \end{aligned}$$

where  $p(\mu)$  is the characteristic equation of the matrix

$$J_{T^2}(\Phi, \Psi) = \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}.$$

By Lemma 17, suppose that  $\{\phi_1, \psi_1\}$  is a minimal period-two solution such that  $\{\phi_1, \psi_1\}$  are roots of multiplicity two of (13) and  $\{\phi_2, \psi_2\}$  are roots of multiplicity one of (13). In view of Lemma 7 [2] I have  $y'_{\bar{F}}(\phi_1) - y'_{\bar{G}}(\phi_1) = 0$  and  $y'_{\bar{F}}(\phi_2) - y'_{\bar{G}}(\phi_2) \neq 0$  which implies that  $\{\phi_1, \psi_1\}$  is non-hyperbolic. Since  $\det J_T(\phi_2, \psi_2) > 0$  and  $\text{tr} J_T(\phi_2, \psi_2) > 0$  I have that  $\{\phi_2, \psi_2\}$  is hyperbolic. Since  $E_{SW}$  is a repeller it must be that  $\{\phi_2, \psi_2\}$  is a saddle point. If  $\{\phi_1, \psi_1\}$  is non-hyperbolic of the stable type then the rest of the proof follows from Theorems 29.

□

### 1.4.2 Example 2: $x_{n+1} = Ax_n + \frac{Bx_{n-1}^2}{1+x_{n-1}^2}$

In this part I apply Theorems 26-30 to describe the global dynamics of difference equation in the title.

#### The equilibrium points

I consider the difference equation

$$x_{n+1} = Ax_n + \frac{Bx_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \dots \quad (15)$$

where  $A, B > 0$  and the initial conditions  $x_{-1}, x_0$  are non-negative. In view of the above restriction on the initial conditions of equation (148), the equilibrium points of equation (148) are the positive solutions of the equation

$$\bar{x} = A\bar{x} + \frac{B\bar{x}^2}{1+\bar{x}^2}$$

or equivalently

$$(A-1)x^3 + Bx^2 + (A-1)x = 0. \quad (16)$$

Now I have the following result.

**Lemma 9** *The equilibrium points of equation (148) satisfy:*

- i) *If  $A \geq 1$  or  $B^2 - 4(A-1)^2 < 0$  then equation (148) has a unique equilibrium point  $x_0 = 0$ ;*
- ii) *If  $A < 1$  and  $B^2 - 4(A-1)^2 = 0$  then equation (148) has two equilibrium points  $x_0 = 0, x_{SW} = \frac{B}{2(1-A)}$ ;*
- iii) *If  $A < 1$  and  $B^2 - 4(A-1)^2 > 0$  then equation (148) has three equilibrium points  $x_0 = 0, x_{SW} = \frac{B - \sqrt{B^2 - 4(A-1)^2}}{2(1-A)}$  and  $x_{NE} = \frac{B + \sqrt{B^2 - 4(A-1)^2}}{2(1-A)}$ .*

Next, I investigate the stability of the equilibrium points of equation (148).

Set

$$f(u, v) = Au + \frac{Bv^2}{1+v^2}$$

and observe that  $f_u(u, v) > 0$  and  $f_v(u, v) > 0$ . The next three lemmas are straightforward.

**Lemma 10** *Assume that  $A, B > 0$ . Then the equilibrium point  $x_0$  of equation (148) is locally asymptotically stable if  $A < 1$ , non-hyperbolic if  $A = 1$  and a saddle point if  $A > 1$ .*

**Lemma 11** *Assume that  $A < 1$ . If  $B^2 - 4(A - 1)^2 > 0$  then the equilibrium point equation (148)  $x_{NE}$  of equation (148) is locally asymptotically stable and non-hyperbolic if  $B^2 - 4(A - 1)^2 = 0$ .*

**Proof.** The proof follows from Theorem 1.1.1 in [54] and the fact that

$$1 - p - q = \frac{(1 - A)\sqrt{B^2 - 4(A - 1)^2}}{B}, \text{ and } p, q > 0,$$

where  $p = f_u(x_{NE}, x_{NE})$  and  $q = f_v(x_{NE}, x_{NE})$ . □

**Lemma 12** *Assume that  $A < 1$ . Then the equilibrium point  $x_{SW}$  is:*

- i) a saddle point if  $(A - 1)\sqrt{B^2 - 4(A - 1)^2} + 2AB > 0$  and  $B^2 - 4(A - 1)^2 > 0$ ;*
- ii) a repeller if  $(A - 1)\sqrt{B^2 - 4(A - 1)^2} + 2AB < 0$  and  $B^2 - 4(A - 1)^2 > 0$ ;*
- iii) a non-hyperbolic point if  $B^2 - 4(A - 1)^2 = 0$ .*

**Proof.** The proof follows from Theorem 1.1.1 in [54] and the fact that

$$1 - p - q = \frac{(A - 1)\sqrt{B^2 - 4(A - 1)^2}}{B} \text{ and } 1 + p - q = \frac{(A - 1)\sqrt{B^2 - 4(A - 1)^2} + 2AB}{B}$$

where  $p = f_u(x_{SW}, x_{SW})$  and  $q = f_v(x_{SW}, x_{SW})$ . □

### Period-two solutions

Next, I investigate the existence and stability of the positive minimal period-two solutions of equation(148).

Let  $\{\phi, \psi\}$  be a minimal period-two solution of equation (148). Then

$$\phi = A\psi + \frac{B\phi^2}{1 + \phi^2}, \quad \psi = A\phi + \frac{B\psi^2}{1 + \psi^2}, \quad \phi \neq \psi$$

which is true if and only if  $\phi \neq \psi$  and

$$-A\psi - A\psi\phi^2 - B\phi^2 + \phi^3 + \phi = 0, \quad (17)$$

$$-A\psi^2\phi - A\phi - B\psi^2 + \psi^3 + \psi = 0. \quad (18)$$

By eliminating  $\psi$  from (17) and (18) I obtain

$$\phi (A\phi^2 + A + B\phi - \phi^2 - 1) \tilde{g}(\phi) = 0$$

and by eliminating  $\phi$  from (17) and (18) I obtain

$$\psi (A\psi^2 + A + B\psi - \psi^2 - 1) \tilde{g}(\psi) = 0$$

where

$$\begin{aligned} \tilde{g}(x) = & (A+1)x^6 - 2(A+1)Bx^5 + (A+1)x^4 (A^2 + B^2 + 2) - (A^2 + 3A + 2) Bx^3 \\ & + x^2 (2A^3 + 2A^2 + AB^2 + A + 1) - A(A+1)Bx + A^2(A+1). \end{aligned}$$

Since  $(A-1)x^2 + Bx + A-1 \neq 0$  for any  $x$  different from the equilibrium point, the minimal period-two solutions are solutions of the equation

$$\tilde{g}(x) = 0 \quad (19)$$

The proof of the following lemma is similar to the proof of Lemma 17, so I skip it.

**Lemma 13** *Let*

$$\begin{aligned}
\Delta &= 4A^5 + 20A^4 + 8A^3B^2 + 40A^3 - 12A^2B^2 + 40A^2 + 4AB^4 - 21AB^2 + 20A - B^2 + 4, \\
\Delta_1 &= -(3A^2 - 2B^2 + 6), \\
\Delta_2 &= -(8A^7 + 8A^6 + 16A^5B^2 - 32A^4B^2 + 8A^3B^4 - 45A^3B^2 \\
&\quad - 24A^3 + 39A^2B^2 - 24A^2 + 8AB^4 - 40AB^2 + 16A - 4B^2 + 16), \\
\Delta_3 &= 64A^{11} + 192A^{10} + 212A^9B^2 - 64A^9 + 12A^8B^2 - 704A^8 + 264A^7B^4 + 772A^7B^2 - 384A^7 \\
&\quad - 108A^6B^4 + 2828A^6B^2 + 896A^6 + 148A^5B^6 + 247A^5B^4 + 1516A^5B^2 + 896A^5 + 8A^4B^6 \\
&\quad + 237A^4B^4 - 572A^4B^2 - 384A^4 + 32A^3B^8 + 60A^3B^6 - 363A^3B^4 + 140A^3B^2 - 704A^3 \\
&\quad + 32A^2B^8 - 224A^2B^6 + 247A^2B^4 + 100A^2B^2 - 64A^2 - 40AB^6 + 240AB^4 - 336AB^2 \\
&\quad + 192A + 12B^4 - 64B^2 + 64, \\
\Delta_4 &= -(8 - 32A^2 + 48A^4 - 32A^6 + 8A^8 - 10B^2 - 30AB^2 + 57A^2B^2 \\
&\quad + 8A^3B^2 + 8A^4B^2 + 70A^5B^2 - 7A^6B^2 + 2B^4 + 26AB^4 - 17A^2B^4 + 8A^3B^4 \\
&\quad - 15A^4B^4 - 4AB^6 - 4A^2B^6), \\
\Delta_5 &= -(4A^4 - 16A^3 + 3A^2B^2 + 24A^2 + 2AB^2 - 16A - B^2 + 4).
\end{aligned}$$

*Then the following holds:*

- a) *Consider equation (19). Then, all its real roots are positive numbers. Furthermore, equation (148) has up to three minimal period-two solutions.*
- b) *If  $\Delta_i > 0$ , for all  $1 \leq i \leq 5$  and  $\Delta > 0$  then equation (19) has six real roots. Consequently, equation (148) has three minimal period-two solutions.*
- c) *If  $\Delta_i \leq 0$  for some  $1 \leq i \leq 4$  and  $\Delta_5 > 0, \Delta > 0$  then equation (19) has two distinct real roots and two pairs of conjugate imaginary roots. Consequently, equation (148) has one minimal period-two solutions.*
- d) *If  $\Delta_i > 0$  for all  $1 \leq i \leq j-1$  and  $\Delta_i < 0$  for all  $j \leq i \leq 4$  for some  $1 \leq j \leq 5$  and  $\Delta_5 < 0, \Delta > 0$  then equation (19) has four distinct real roots*



and one pair of conjugate imaginary roots. Consequently, equation (148) has two minimal period-two solutions.

e) If  $\Delta_i \leq 0, \Delta_{i+1} \geq 0$  for some  $1 \leq i \leq 3$  and  $\Delta_5 < 0, \Delta > 0$  then equation (19) has three pairs of conjugate imaginary roots. Consequently, equation (148) has no minimal period-two solution.

f) Assume that  $\Delta = 0, \Delta_3 \neq 0, \Delta_5 \neq 0$ .

f.1) If  $(\Delta_1 \leq 0, \Delta_2 \geq 0)$  or  $(\Delta_2 \leq 0, \Delta_3 > 0)$  then equation (19) has no real roots and has two distinct pairs of conjugate imaginary roots one of them is of multiplicity one and the other one of multiplicity two. Consequently, equation (148) has no minimal period-two solutions.

f.2) If  $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$  then equation (19) has four distinct real roots, two of them are multiplicity two and the other two of multiplicity one and has no conjugate imaginary roots. Consequently, equation (148) has two minimal period-two solutions.

## The global behavior

In this section I describe the global behavior of equation (148). Equation (148) has three equilibrium points  $\bar{x}_0, \bar{x}_{SW}, \bar{x}_{NE} \in I$  such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is unstable. One can see that all minimal period two solutions of (148) belong to  $\text{int}(Q_2(E_{SW}) \cup Q_4(E_{SW}))$ .

Let  $u_{n+1} = Au_n + B$ . By mathematical induction, it is easy to see that  $x_n \leq u_n$  for  $n > 0$  if  $x_0 \leq u_0$ . Since

$$u_n = u_0 A^n + \frac{B(1 - A^n)}{1 - A}, A \neq 1$$

and

$$u_n = u_0 + nB, A = 1$$

I obtain that  $u_n \rightarrow \frac{B}{1-A}$  as  $n \rightarrow \infty$  if  $A < 1$ . Thus, I conclude that the interval  $[0, \frac{B}{1-A} + \epsilon]$  where  $\epsilon > 0$ , attracts all solutions, when  $A < 1$ .

**Lemma 14** *If  $A \geq 1$  then there exists a unique equilibrium point  $x_0 = 0$  and every solution  $\{x_n\}$  satisfies*

$$\lim_{n \rightarrow \infty} x_n = \infty.$$

**Proof.** By using the difference inequalities method [11], the proof follows from the fact that  $x_n \geq v_n$  for  $n > 0$  where  $x_0 = v_0$  and  $v_n = Av_{n-1}$  for  $n > 1$ .  $\square$

**Theorem 12** *Assume that  $A < 1$  and  $B^2 - 4(A - 1)^2 > 0$ . Then the following holds:*

- i) If  $\Delta_i \leq 0, \Delta_{i+1} \geq 0$  for some  $1 \leq i \leq 3$  and  $\Delta_5 < 0, \Delta > 0$  then equation (148) has three equilibrium point such that  $0 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is a saddle point and has no period-two solution. The global behavior of equation (148) is described by Theorem 1. For example, this happens for  $A = 0.3$  and  $B = 2.0$ .*
- ii) If  $\Delta_i \leq 0$  for some  $1 \leq i \leq 4$  and  $\Delta_5 > 0, \Delta > 0$  then equation (148) has three equilibrium points such that  $0 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is a repeller and one period-two solution  $\{\phi_1, \psi_1\}$  which is a saddle point. The global behavior of equation (148) is described by Theorem 27. For example, this happens for  $A = 0.1$  and  $B = 1.9$ .*
- iii) If  $\Delta_i > 0$ , for all  $1 \leq i \leq 5$  and  $\Delta > 0$  then equation (148) has three equilibrium points such that  $0 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally*

asymptotically stable and  $x_{SW}$  is a repeller and three minimal period-two solutions  $\{\phi_1, \psi_1\}$ ,  $\{\phi_2, \psi_2\}$  and  $\{\phi_3, \psi_3\}$  such that  $(\phi_1, \psi_1) \preceq_{ne} (\phi_2, \psi_2) \preceq_{ne} (\phi_3, \psi_3)$  where  $\{\phi_1, \psi_1\}$ , and  $\{\phi_3, \psi_3\}$  are saddle points and  $\{\phi_2, \psi_2\}$  is locally asymptotically stable. The global behavior of equation (148) is described by Theorem 28. For example, this happens for  $A = 0.1$  and  $B = 2.0$ .

iv) If  $\Delta = 0$  and  $\Delta_5 > 0$  and  $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$  then equation (148) has three equilibrium points such that  $0 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where  $x_0$  and  $x_{NE}$  are locally asymptotically stable and  $x_{SW}$  is a repeller and two minimal period-two solutions  $\{\phi_1, \psi_1\}$  and  $\{\phi_2, \psi_2\}$  such that  $(\phi_1, \psi_1) \preceq_{ne} (\phi_2, \psi_2)$  where either  $\{\phi_1, \psi_1\}$  is non-hyperbolic and  $\{\phi_2, \psi_2\}$  is a saddle point or  $\{\phi_1, \psi_1\}$  is a saddle point and  $\{\phi_2, \psi_2\}$  is non-hyperbolic. If period-two solution which is non-hyperbolic is of stable type then the global behavior of equation (148) is described by Theorems 28 and 29. For example, this happens for  $A = 0.1$  and  $B = 1.97282$ .

**Remark 2** Similarly as in the previous two examples, one can see that the difference equation

$$x_{n+1} = \frac{Ax_n}{1+x_n} + \frac{Bx_{n-1}^2}{1+x_{n-1}^2}, \quad A, B > 0, x_{-1}, x_0 \geq 0, \quad n = 0, 1, \dots$$

has three equilibrium points  $\bar{x}_0, \bar{x}_{SW}, \bar{x}_{NE} \in [0, \infty)$  such that  $0 = \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$  where the equilibrium points  $\bar{x}_0$  and  $\bar{x}_{NE}$  are locally asymptotically stable and  $\bar{x}_{SW}$  is unstable and  $x_n < A + B$  for all  $n \geq 1$ . Further, one can see that all its minimal period two solutions belong to  $\text{int}(Q_2(E_{SW}) \cup Q_4(E_{SW}))$ . It has up to the three minimal period-two solutions and global behavior can be described by Theorem 12, where determinants  $\Delta_i$  will have different values which depend in the discrimination matrix.

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**MANUSCRIPT 2**

**Global Attractivity for Nonautonomous Difference Equation via  
Linearization**

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## 2.1 Introduction and preliminaries

Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, \dots, \vec{x}_{n-k}), \quad n = 0, 1, \dots \quad (20)$$

where  $k \in \{0, 1, \dots\}$  and the initial conditions are real vectors in  $\mathbb{R}^p, p \geq 2$ . In many cases I investigate equation(20) by embedding equation(20) into a higher iteration of the form

$$\vec{x}_{n+l} = F(n, \vec{x}_{n+l-1}, \dots, \vec{x}_{n-k}), \quad n = 0, 1, \dots \quad (21)$$

where  $l \in \{1, 2, \dots\}$ , see [4, 6, 9]. By linearizing equation (21) and bringing it to the form

$$\vec{x}_{n+1} = \sum_{i=1-l}^k g_i \vec{x}_{n-i}, \quad (22)$$

where  $g_i$  in general, depend on  $n$  and the state variables  $\vec{x}_k$  I can prove very general attractivity and asymptotic stability results for both autonomous and nonautonomous difference equations. The functions  $g_i$  are in general matrices but they can also be the scalars as well, see Section 2.3. This approach was used to get effective and applicable global asymptotic and global attractivity results for linear fractional difference equation, see [2] and quadratic fractional difference equation, see [3] with both constant and nonconstant coefficients. Furthermore, this approach produced global asymptotic and global attractivity results for nonautonomous difference equations with very general coefficients which can be discontinuous functions of  $n$  or state variables, see [4, 6, 9]. See [1, 8, 63, 12] for the case of monotone systems, where more precise results are obtained.

In this paper I use method of linearization to extend some of the results about the global attractivity and asymptotic stability of scalar equation from [4] to the case of vector equation (21). I illustrate our results with many examples that include some transition functions from mathematical biology such as linear,

Beverton-Holt, sigmoid Beverton-Holt, etc., see [49, 8, 60, 12, 79] for related results. The rest of this section contains some definitions and preliminary results. Second section contains our main results on global attractivity in the case when the sum of the norms of  $g_i$  is less than 1. The third section gives some results on global attractivity in the delicate case when the sum of the scalar functions  $g_i$  is 1. The fourth section provides several examples which illustrate our results.

Denote by  $\|\vec{x}\|$  any norm in  $\mathbb{R}^p$ .

**Definition 1** *The zero equilibrium of equation (22) is stable if for  $(\forall \epsilon > 0)(\exists \delta > 0, N)$ :*

$$\|\vec{x}_i\| < \delta, i = -k, \dots, 0 \implies \|\vec{x}_n\| < \epsilon, \text{ for all } n \geq N.$$

*The zero equilibrium is asymptotically stable if it is stable and*

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{0}.$$

**Lemma 15** *Let  $\mathbf{I} - \sum_{i=0}^k g_i$  be invertible for  $n = 1, 2, \dots$ , where  $\mathbf{I}$  is identity matrix. Then equation (22) has no nonzero equilibrium.*

**Proof.** Otherwise, equation (22) has the equilibrium  $\bar{\mathbf{x}} \neq \vec{0}$ . By plugging  $\vec{x}_n = \bar{\mathbf{x}}$  in equation (22) I get

$$(\mathbf{I} - \sum_{i=0}^k g_i) \bar{\mathbf{x}} = \vec{0},$$

which implies  $\bar{\mathbf{x}} = \vec{0}$ , which is a contradiction.  $\square$

**Remark 3** The matrix  $\mathbf{I} - \sum_{i=0}^k g_i$  is invertible if the condition

$$\left\| \sum_{i=0}^k g_i \right\| < 1 \tag{23}$$

is satisfied in which case I have

$$(\mathbf{I} - \sum_{i=0}^k g_i)^{-1} = \sum_{k=0}^{\infty} \sum_{i=0}^k g_i. \tag{24}$$



The condition (23) is implied by more applicable condition

$$\sum_{i=0}^k \|g_i\| < 1. \quad (25)$$

**Remark 4** Equation (20) admits the following generalized identity

$$\vec{x}_{n+1} - \sum_{i=0}^k g_i \vec{K} = \sum_{i=0}^k g_i (\vec{x}_{n-i} - \vec{K}), \quad (26)$$

where  $\vec{K}$  is an arbitrary vector. Generalized identity (26) implies

$$\|\vec{x}_{n+1} - \sum_{i=0}^k g_i \vec{K}\| \leq \sum_{i=0}^k \|g_i\| \|\vec{x}_{n-i} - \vec{K}\|. \quad (27)$$

Furthermore by taking  $\vec{K} = \vec{0}$  in equation (27), I obtain another useful inequality

$$\|\vec{x}_{n+1}\| - L \sum_{i=0}^k \|g_i\| \leq \sum_{i=0}^k \|g_i\| (\|\vec{x}_{n-i}\| - L), \quad (28)$$

where  $L$  is an arbitrary constant.

**Lemma 16** Suppose that equation (20) has the linearization (22) and the functions  $g_i : R^{p+1} \rightarrow M_{p \times p}$ , where  $M_{p \times p}, p \geq 1$  is the set of all real  $p \times p$  matrices, are such that

$$\sum_{i=0}^k \|g_i\| \leq 1, \quad n = 0, 1, \dots$$

Then if equation (20) has the zero equilibrium it is a stable fixed point.

**Proof.** Assume that equation (20) has the zero equilibrium and the linearization (22). By taking  $\vec{K} = \vec{0}$  in equation (27) I have

$$\|\vec{x}_{n+1}\| \leq \sum_{i=0}^k \|g_i\| \|\vec{x}_{n-i}\|.$$

Assume that  $\sum_{i=0}^k \|\vec{x}_{-i}\| < \delta$ . Take  $\delta = \epsilon$ . Then  $\|\vec{x}_{-i}\| < \delta$  for  $i = 0, 1, \dots$ . Hence

$$\|\vec{x}_1\| \leq \sum_{i=0}^k \|g_i\| \|\vec{x}_{-i}\| < \delta \sum_{i=0}^k \|g_i\| \leq \delta = \epsilon,$$

$$\|\vec{x}_2\| \leq \sum_{i=0}^k \|g_i\| \|\vec{x}_{1-i}\| < \delta \sum_{i=0}^k \|g_i\| \leq \delta = \epsilon$$

and so by induction  $\|\vec{x}_n\| < \epsilon$  for  $n \geq -k$ . □

## 2.2 Main results

In this section I present our main results on global attractivity and global asymptotic stability of the equilibrium solutions of equation (20) which has the linearization (22).

**Theorem 13** *Let  $l \in \{1, 2, \dots\}$ . Suppose that equation (20) has the linearization (22) subject to the condition*

$$\sum_{i=1-l}^k \|g_i\| \leq 1, n = 0, 1, \dots \quad (29)$$

*Let  $M_0 = \max\{\|\vec{x}_{l-1}\|, \dots, \|\vec{x}_{-k}\|\}$ . Then every solution of equation (20) is bounded. In particular  $\|\vec{x}_n\| \leq M_0$  for  $n \geq -k$ .*

**Proof.** Let  $L \in R$ . Then equation (28) implies

$$\|\vec{x}_{n+l}\| - L \sum_{i=1-l}^k \|g_i\| \leq \sum_{i=1-l}^k \|g_i\| (\|\vec{x}_{n-i}\| - L), \quad n = 0, 1, \dots \quad (30)$$

By taking  $L = M_0$  and  $n = 0$  in equation (30), I obtain

$$\|\vec{x}_l\| - M_0 \sum_{i=1-l}^k \|g_i\| \leq \|g_{1-l}\| (\|\vec{x}_{l-1}\| - M_0) + \dots + \|g_k\| (\|\vec{x}_{-k}\| - M_0) \leq 0,$$

which in view of equation (29) implies  $\|\vec{x}_l\| \leq M_0$ . By using induction, I obtain

$$\|\vec{x}_{n+l}\| - M_0 \sum_{i=1-l}^k \|g_i\| \leq \|g_{1-l}\| (\|\vec{x}_{n+l-1}\| - M_0) + \dots + \|g_k\| (\|\vec{x}_{n-k}\| - M_0) \leq 0, \quad n = 0, 1, \dots$$

and so

$$\|\vec{x}_{n+l}\| \leq M_0 \sum_{i=1-l}^k \|g_i\| \leq M_0, \quad n = 0, 1, \dots$$

Thus  $\|\vec{x}_{n+l}\| \leq M_0$  for  $n \geq -k$ . □

**Theorem 14** *Let  $l \in \{1, 2, \dots\}$ . Suppose that equation (20) has the linearization (22) where the functions  $g_i : R^{k+1} \rightarrow M_{p \times p}$  are such that*

$$\sum_{i=1-l}^k \|g_i\| \leq a < 1, \quad n = 0, 1, \dots \quad (31)$$

Then

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{0}.$$

**Proof.** Let  $L \in R$ . Then every solution of equation (22) satisfies the inequality (30). Let  $\gamma = l + k$ . Define  $M_N = \max\{\|\vec{x}_{\gamma N+l-1}\|, \dots, \|\vec{x}_{\gamma N-k}\|\}$  for  $N = 0, 1, \dots$ . Observe that if  $\|\vec{x}_{\gamma N+l-1}\| = \dots = \|\vec{x}_{\gamma N-k}\| = \vec{0}$  for some  $N \geq 0$ , then by (30) with  $L = 0$  I get that

$$\|\vec{x}_{\gamma N+l+j}\| = \vec{0}, \quad j = 0, 1, \dots$$

and so  $\lim_{n \rightarrow \infty} \vec{x}_n = \vec{0}$ .

Assume that  $M_N > 0$  for all  $N \geq 0$ . By using (30) with  $L = M_N$  and  $n = \gamma N$  I obtain

$$\|\vec{x}_{\gamma N+l}\| - \sum_{i=1-l}^k \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l-1}\| - M_N) + \dots + \|g_k\|(\|\vec{x}_{\gamma N-k}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l}\| \leq \sum_{i=1-l}^k \|g_i\| M_N \leq a M_N < M_N.$$

Similarly, by taking  $n = \gamma N + 1$  in (30) I obtain

$$\|\vec{x}_{\gamma N+l+1}\| - \sum_{i=1-l}^k \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l}\| - M_N) + \dots + \|g_k\|(\|\vec{x}_{\gamma N-k+1}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l+1}\| \leq \sum_{i=1-l}^k \|g_i\| M_N \leq a M_N < M_N.$$

Hence by induction I have that

$$\|\vec{x}_{\gamma N+l+j}\| \leq \sum_{i=1-l}^k \|g_i\| M_N \leq a M_N < M_N.$$

Thus

$$M_{N+1} \leq a M_N < M_N, \tag{32}$$

and so the sequence  $\{M_N\}_{N=0}^{\infty}$  is decreasing sequence bounded below by zero. Furthermore (32) implies

$$M_N \leq a^{N+1} M_0 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$

Hence

$$0 \leq \lim_{N \rightarrow \infty} \vec{x}_{\gamma_{N-j}} \leq \lim_{N \rightarrow \infty} M_N = 0, \quad j = 1 - l, \dots, k.$$

Therefore

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{0}.$$

□

**Corollary 2** *Suppose that equation (20) has the linearization (22), where  $l = 1$  and the functions  $g_i : R^{k+1} \rightarrow M_{p \times p}$  are such that*

$$\sum_{i=0}^k \|g_i\| \leq a < 1, \quad n = 0, 1, \dots$$

*Then if equation (20) has a zero equilibrium it is globally asymptotically stable.*

Assuming that  $f$  is differentiable in some neighborhood of the equilibrium solution  $\bar{x}$ , by applying Theorem 14 and Lemma 16 to the standard linearization of equation (20) about the equilibrium solution  $\bar{x}$

$$\vec{x}_{n+1} = \sum_{i=0}^k \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \dots, \bar{x}) \vec{x}_{n-i}, \quad n = 0, 1, \dots, \quad (33)$$

where  $\frac{\partial f}{\partial x_{n-i}}(\bar{x}, \dots, \bar{x})$  is the Jacobian matrix evaluated at the equilibrium point, I obtain the following result, which is local in the nature because of the fact that the standard linearization is local.

**Corollary 3** *Assume that  $f$  is differentiable in some neighborhood of the equilibrium solution  $\bar{x}$ . The equilibrium  $\bar{x}$  of equation (20) is locally asymptotically stable if*

$$\sum_{i=0}^k \left\| \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \dots, \bar{x}) \right\| \leq a < 1.$$

### 2.3 The case when $g_i$ are scalar functions

In this section I consider the case when all  $g_i$  are scalar functions. In this case the linearization (22) is equivalent to  $p$  scalar equations of the form

$$x_{n+1}^m = \sum_{i=1-l}^k g_i x_{n-i}^m, \quad n = 0, 1, \dots; m = 1, \dots, p. \quad (34)$$

For instance, in the case of second order difference equation in  $\mathbb{R}^2$ , I have that vector equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = g_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + g_1 \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \quad n = 0, 1, \dots \quad g_0, g_1 \geq 0 \quad (35)$$

is equivalent to the system

$$x_{n+1} = g_0 x_n + g_1 x_{n-1} \quad (36)$$

$$y_{n+1} = g_0 y_n + g_1 y_{n-1}.$$

The next results apply to a special linearization (22) of equation (20), where all  $g_i$  are scalar functions.

**Theorem 15** *Let  $l \in \{1, 2, \dots\}$ . Suppose that equation (20) has the linearization (22), where the functions  $g_i : \mathbb{R}^{k+1} \rightarrow [0, \infty)$  are such that*

$$\sum_{i=1-l}^k g_i \geq a > 1, \quad n \geq 0.$$

*Then if for some  $n \geq 0$*

*(a)  $\vec{x}_{n+l-1}, \dots, \vec{x}_{n-k} > 0$ , then  $\lim_{n \rightarrow \infty} \vec{x}_n = \infty$ , componentwise;*

*(b)  $\vec{x}_{n+l-1}, \dots, \vec{x}_{n-k} < 0$ , then  $\lim_{n \rightarrow \infty} \vec{x}_n = -\infty$ , componentwise.*

**Proof.** Proof follows from Theorem 2 in [4] applied to equation(34).  $\square$

A delicate case when

$$\sum_{i=1-l}^k g_i = 1, \quad n = 0, 1, \dots \quad (37)$$

is treated in the following three results.

**Theorem 16** *Suppose that on some interval  $I$  equation (20) has the linearization (22), where the functions  $g_i : \mathbb{R}^{k+1} \rightarrow [0, \infty)$  are such that (37) is satisfied. Then there exists  $A > 0$  such that for  $n \geq 0$  every positive  $g_i$  satisfies*

$$A \leq g_i \leq 1, \quad n = 0, 1, \dots \quad (38)$$

**Proof.** Proof follows from Proposition 3 in [4] applied to equation (34).  $\square$

**Theorem 17** *Suppose that on some interval  $I$  equation (20) has the linearization (22), where the functions  $g_i : \mathbb{R}^{k+1} \rightarrow [0, \infty)$  are such that (37) is satisfied. Assume that there exists  $A > 0$  such that*

$$g_{1-l} \geq A, \quad n = 0, 1, \dots \quad (39)$$

*Then if  $\vec{x}_{-k}, \dots, \vec{x}_0 \in I$*

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{L},$$

*where  $\vec{L} \in I^p$  is a constant vector*

**Proof.** Proof follows from Theorem 4 in [4] applied to equation (34).  $\square$

**Theorem 18** *Suppose that on some interval  $I \subset \mathbb{R}$  equation (20) has the linearization (22), where the functions  $g_i : \mathbb{R}^{k+1} \rightarrow [0, \infty)$  are such that (37) is satisfied. Assume that there exists  $A > 0$  such that for some  $j \in \{2-l, \dots, k-1\}$*

$$g_j \geq A, g_{j+1} \geq A, \quad n = 0, 1, \dots \quad (40)$$

*If  $\vec{x}_{l-1}, \dots, \vec{x}_{-k} \in I$ , then*

$$\lim_{n \rightarrow \infty} \vec{x}_n = \vec{L},$$

*where  $\vec{L} \in I^p$  is a constant vector*

**Proof.** Proof follows from Theorem 5 in [4] applied to equation (34).  $\square$

## 2.4 Examples

In this section I present some examples that illustrate our results.

**Example 1** Every solution of the vector equation in  $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots,$$

where  $a, d > 0, b_n, c_n \geq 0, x_0, y_0 \geq 0, n = 0, 1, \dots$ , converges to the zero equilibrium if  $\max\{a + U_c, d + U_b\} < 1$  is satisfied, where  $U_b$  and  $U_c$  are upper bounds of sequences  $\{b_n\}$  and  $\{c_n\}$  respectively. Indeed, in this case if  $\|x\|$  denotes the  $L_1$  norm I have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max \left\{ \frac{a}{1+x_n} + c_n, \frac{d}{1+y_n} + b_n \right\} \leq \max\{a+U_c, d+U_b\} < 1,$$

that is  $U_c < 1 - a, U_b < 1 - d$ , and the result follows from Theorem 14 and Corollary 2. Thus in this case the zero equilibrium is globally asymptotically stable. If I use  $L_2$  norm I have that the zero equilibrium is globally asymptotically stable if  $\max\{a + U_b, d + U_c\} < 1$  is satisfied.

**Example 2** Every solution of the vector equation in  $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots, \quad (41)$$

where  $a, b, c, d > 0, x_0, y_0 \geq 0$ , converges to the zero equilibrium if  $\max\{a + c, b + d\} < 1$  is satisfied. Indeed, in this case if  $\|x\|$  denotes the  $L_1$  norm I have that

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max \left\{ \frac{a}{1+x_n} + c, \frac{d}{1+y_n} + b \right\} \leq \max\{a + c, b + d\} < 1$$

and the result follows from Theorem 14 and Corollary 2. Thus in this case the zero equilibrium is globally asymptotically stable. If I use  $L_2$  norm I have that  $\max\{a + b, c + d\} < 1$  implies that the zero equilibrium is globally asymptotically stable.

Next, consider the positive equilibrium  $E(\bar{x}, \bar{y})$ . Then I have that the positive equilibrium  $E(\bar{x}, \bar{y})$  of system (41) satisfies the system

$$\begin{aligned}\bar{x} &= a \frac{\bar{x}}{1+\bar{x}} + b\bar{y} \\ \bar{y} &= c\bar{x} + d \frac{\bar{y}}{1+\bar{y}}.\end{aligned}\tag{42}$$

which implies

$$\begin{aligned}\bar{x} \frac{1+\bar{x}-a}{1+\bar{x}} &= b\bar{y} \\ \bar{y} \frac{1+\bar{y}-d}{1+\bar{y}} &= c\bar{x}.\end{aligned}$$

Thus the positive equilibrium exists if

$$\bar{x} > a - 1, \bar{y} > d - 1.\tag{43}$$

Linearizing system (41) about the positive equilibrium  $E$  gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & b \\ c & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \dots,\tag{44}$$

where  $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$ . By using Theorem 14 and Corollary 2 with  $L_1$  norm, I obtain that the condition for global asymptotic stability of  $E(\bar{x}, \bar{y})$  to be

$$\bar{x} > \frac{a+c-1}{1-c} \quad \text{if } c < 1 < a+c, \quad \bar{y} > \frac{b+d-1}{1-b} \quad \text{if } b < 1 < b+d.$$

If I use  $L_2$  norm I obtain sufficient condition for global asymptotic stability of  $E(\bar{x}, \bar{y})$  to be

$$\bar{x} > \frac{a+b-1}{1-b} \quad \text{if } b < 1 < a+b, \quad \bar{y} > \frac{c+d-1}{1-c} \quad \text{if } c < 1 < c+d.$$

**Example 3** Every solution of the vector equation in  $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots,\tag{45}$$

where  $a, b, c, d > 0, x_0, y_0 \geq 0, n = 0, 1, \dots$ , converges to the zero equilibrium if  $\max\{a+c, b+d\} < 1$  is satisfied. Indeed, in this case if  $\|x\|_1$  denotes the  $L_1$  norm I have

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_1 = \max \left\{ \frac{a}{1+x_n} + \frac{c}{1+x_n}, \frac{b}{1+y_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+c, b+d\} < 1$$



and the result follows from Theorem 14 and Corollary 2. Thus in this case the zero equilibrium is globally asymptotically stable.

In the case if  $\|x\|_2$  denotes the  $L_2$  norm I have

$$\begin{aligned}\|g_0\|_2 &= \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_2 \\ &= \max \left\{ \frac{a}{1+x_n} + \frac{b}{1+y_n}, \frac{c}{1+x_n} + \frac{d}{1+y_n} \right\} \\ &\leq \max\{a+b, c+d\} \\ &< 1.\end{aligned}$$

In this case the condition for global asymptotic stability of the zero equilibrium becomes  $\max\{a+b, c+d\} < 1$ .

Now, consider global attractivity of the positive equilibrium  $E(\bar{x}, \bar{y})$  of system (45). The positive equilibrium of system (45) satisfies the system

$$\begin{aligned}\bar{x} &= a \frac{\bar{x}}{1+\bar{x}} + b \frac{\bar{y}}{1+\bar{y}} \\ \bar{y} &= c \frac{\bar{x}}{1+\bar{x}} + d \frac{\bar{y}}{1+\bar{y}}.\end{aligned}\tag{46}$$

Adding two equations in (46) I obtain

$$\bar{x} + \bar{y} = (a+c) \frac{\bar{x}}{1+\bar{x}} + (b+d) \frac{\bar{y}}{1+\bar{y}},$$

which implies

$$\frac{\bar{x}}{1+\bar{x}}(1+\bar{x}-a-c) = \frac{\bar{y}}{1+\bar{y}}(b+d-1-\bar{y})$$

and so I obtain that the positive equilibrium satisfies

$$\bar{x} > a+c-1 \Leftrightarrow \bar{y} < b+d-1.\tag{47}$$

Linearizing system (45) about the positive equilibrium  $E$  gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \dots,$$

where  $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$ . By using Theorem 14 and Corollary 2 with  $L_1$  norm, I obtain that the condition

$$\bar{x} > a+c-1, \bar{y} > b+d-1.\tag{48}$$

is sufficient for the global asymptotic stability of the positive equilibrium solution. The condition (48) contradicts condition (47). If I use  $L_2$  norm I obtain sufficient condition for the global asymptotic stability of the positive equilibrium solution to be

$$\begin{aligned} b\bar{x} + a\bar{y} &< 1 - a - b \\ d\bar{x} + c\bar{y} &< 1 - c - d. \end{aligned}$$

**Example 4** Every solution of the vector equation in  $\mathbb{R}^n$

$$\vec{x}_{n+1} = A_n \vec{x}_n \quad (49)$$

where

$$\vec{x}_n = \begin{bmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^k \end{bmatrix}, \quad A_n = \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix}$$

where  $a_{ij} > 0, i, j = 0, 1, \dots$ ,  $x_0, y_0 \geq 0, n = 0, 1, \dots$ , converges to the zero equilibrium if

$$\begin{aligned} \|g_0\|_1 &= \left\| \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix} \right\|_1 \\ &= \max \left\{ \frac{a_{11}}{1+x_n^1} + \frac{a_{21}}{1+x_n^1} + \cdots + \frac{a_{k1}}{1+x_n^1}, \dots, \frac{a_{1k}}{1+x_n^k} + \frac{a_{2k}}{1+x_n^k} + \cdots + \frac{a_{kk}}{1+x_n^k} \right\} \\ &\leq \max \{ a_{11} + a_{21} + \cdots + a_{k1}, \dots, a_{1k} + a_{2k} + \cdots + a_{kk} \} \\ &= \max_{1 \leq j \leq k} \left\{ \sum_{i=1}^k a_{ij} \right\} < 1, \end{aligned}$$

which follows from Theorem 14 and Corollary 2. Thus in this case the zero equilibrium is globally asymptotically stable.

Now, consider global attractivity of the positive equilibrium of system (49).

The positive equilibrium satisfies the system

$$(A_n(\vec{x}) - \mathbf{I})\vec{x} = \vec{0},$$

where

$$A_n(\vec{x}) = \begin{bmatrix} \frac{a_{11}}{1+\bar{x}^1} & \frac{a_{12}}{1+\bar{x}^2} & \cdots & \frac{a_{1k}}{1+\bar{x}^k} \\ \frac{a_{21}}{1+\bar{x}^1} & \frac{a_{22}}{1+\bar{x}^2} & \cdots & \frac{a_{2k}}{1+\bar{x}^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+\bar{x}^1} & \frac{a_{k2}}{1+\bar{x}^2} & \cdots & \frac{a_{kk}}{1+\bar{x}^k} \end{bmatrix}.$$

Linearizing system (49) about the positive equilibrium  $E$  gives the following system

$$\vec{u}_{n+1} = \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \vec{u}_n, \quad n = 0, 1, \dots,$$

where  $\vec{u}_n = \vec{x}_n - \vec{x}$ . By using Theorem 14 and Corollary 2 with  $L_1$  norm, I obtain that the condition

$$\begin{aligned} \|g_0\|_1 &= \left\| \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \right\|_1 \\ &= \max \left\{ \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} + \cdots + \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)}, \dots, \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} + \cdots + \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \right\} \\ &\leq \max \left\{ \frac{1}{1+\bar{x}} (a_{11} + a_{21} + \cdots + a_{k1}), \dots, \frac{1}{1+\bar{x}} (a_{1k} + a_{2k} + \cdots + a_{kk}) \right\} \\ &= \frac{1}{1+\bar{x}} \max_{1 \leq j \leq k} \left\{ \sum_{i=1}^k a_{ij} \right\} \\ &< 1 \end{aligned}$$

implies the global asymptotic stability of the positive equilibrium solution. By using Theorem 14 and Corollary 2 with  $L_1$  norm, I obtain that the condition for the global asymptotic stability of the positive equilibrium solution is

$$1 + \bar{x} > \sum_{i=1}^k a_{ij} \iff \bar{x} > \sum_{i=1}^k a_{ij} - 1.$$

**Example 5** The cooperative system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad n = 0, 1, \dots, \quad (50)$$

where  $a, b, c, d > 0$ ,  $x_0, y_0 \geq 0$  was considered in [1]. The equilibrium solutions are the zero equilibrium  $E_0(0, 0)$  and when  $a < 1, d < 1$  the unique positive equilibrium solution  $E_+(\bar{x}, \bar{y})$ , is given as

$$\bar{x} = \frac{b}{1-a} \frac{\bar{y}}{1+\bar{y}}, \quad \bar{y} = \frac{bc - (1-d)(1-a)}{(1-d)(b+1-a)},$$

when

$$(1-a)(1-d) < bc. \quad (51)$$

The local stability of system (120) is described with the following result, see [1]

**Claim 1** *Consider system (120).*

- 1.) *The positive equilibrium  $E_+(\bar{x}, \bar{y})$  of system (120) is locally asymptotically stable when (122) holds.*
- 2.) *The zero equilibrium  $E_0(0, 0)$  of system (120) is locally asymptotically stable if  $bc < (1-a)(1-d)$ ; it is a saddle point if  $bc > (1-a)(1-d)$ ; it is a nonhyperbolic equilibrium if  $bc = (1-a)(1-d)$ .*

The global dynamics of system (120) is described with the following result, see [1]:

**Theorem 19** *Consider system (120).*

- 1.) *If  $a \geq 1$  then  $\lim_{n \rightarrow \infty} x_n = \infty$  and  $\lim_{n \rightarrow \infty} y_n = \infty$  if  $d \geq 1$  and  $\lim_{n \rightarrow \infty} y_n = \frac{c}{1-d}$ , if  $d < 1$ .*
- 2.) *If  $d \geq 1$  then  $\lim_{n \rightarrow \infty} y_n = \infty$  and  $\lim_{n \rightarrow \infty} x_n = \infty$  if  $a \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = \frac{b}{1-a}$ , if  $a < 1$ .*
- 3.) *The positive equilibrium  $E_+(\bar{x}, \bar{y})$  of system (120) is globally asymptotically stable when (122) holds.*

4.) The zero equilibrium  $E_+(\bar{x}, \bar{y})$  of system (120) is globally asymptotically stable when  $a < 1, d < 1$  and

$$bc \leq (1-a)(1-d) \quad (52)$$

holds.

Theorem 14 and Corollary 2 implies that any of two conditions  $\max\{a+c, b+d\} < 1$  or  $\max\{a+b, c+d\} < 1$  provides the global asymptotic stability of the zero equilibrium. Both of these conditions imply (125) which is clearly the necessary and sufficient condition for the global asymptotic stability of the zero equilibrium..

Linearizing system (120) about the positive equilibrium  $E(\bar{x}, \bar{y})$  gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & d \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \quad n = 0, 1, \dots,$$

where  $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$ . By using Theorem 14 and Corollary 2 with  $L_1$  or  $L_2$  norm, I obtain that the condition

$$\max \left\{ a + \frac{c}{1+\bar{x}}, \frac{b}{1+\bar{y}} + d \right\} < 1 \quad \text{or} \quad \max \left\{ a + \frac{b}{1+\bar{y}}, \frac{c}{1+\bar{x}} + d \right\} < 1 \quad (53)$$

implies that the positive equilibrium  $E(\bar{x}, \bar{y})$  is globally asymptotically stable. Condition (53) implies condition (122) which is clearly the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium.

**Example 6** Every solution of the vector equation in  $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad n = 0, 1, \dots,$$

where  $a, b, c, d, A, B, C, D > 0, x_{-1}, y_{-1}, x_0, y_0 \geq 0, n = 0, 1, \dots$ , converges to the zero equilibrium if  $\max\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\} + \max\{A+B, \frac{C+D}{2}\} < 1$  is satisfied. Indeed, in this case if  $\|x\|$  denotes the  $L_1$  norm I have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \right\| = \max \left\{ \frac{(a+b)n}{1+n^2}, \frac{(c+d)n}{1+n^3} \right\} \leq \max \left\{ \frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}} \right\}$$

and

$$\|g_1\| = \left\| \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \right\| = \max \left\{ \frac{(A+B)n}{1+n}, \frac{(C+D)n}{1+n^2} \right\} \leq \max \left\{ A+B, \frac{C+D}{2} \right\}$$

and the result follows from Theorem 14 and Corollary 2. Thus in this case the zero equilibrium is globally asymptotically stable.

**Example 7** The vector equation in  $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{ax_n}{1+x_n} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \frac{a}{1+x_n} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0, 1, \dots \quad (54)$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= \frac{ax_n}{1+x_n}x_n + \frac{a}{1+x_n}x_{n-1} \\ y_{n+1} &= \frac{ax_n}{1+x_n}y_n + \frac{a}{1+x_n}y_{n-1}, \quad n = 0, 1, \dots, \end{aligned}$$

where  $a > 0$ . Since  $g_0 + g_1 = a$  for all  $n = 0, 1, \dots$  I have the following result which proof follows from Theorems 14, 15, 17 and Corollary 2.

**Proposition 1** *The following trichotomy holds for equation (54):*

- (a) *if  $a < 1$  then the zero equilibrium of (54) is globally asymptotically stable.*
- (b) *if  $a = 1$  then every nonnegative constant vector  $\vec{L}$  is an equilibrium of (54) and every solution of (54) converges to some constant vector.*
- (a) *if  $a > 1$  then every set of positive (resp. negative) initial conditions generates the solution which component-wise tends to  $\infty$  (resp.  $-\infty$ ).*

Proposition 1 can be extended to the case of corresponding vector equation in  $\mathbb{R}^p$ .

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**MANUSCRIPT 3**

**Global Dynamics of Cooperative Discrete System in the Plane**

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### 3.1 Introduction and Preliminaries

In this paper I consider cooperative system

$$x_{n+1} = ax_n + \frac{by_n^2}{1+y_n^2}, \quad y_{n+1} = \frac{cx_n^2}{1+x_n^2} + dy_n, \quad n = 0, 1, \dots, \quad (55)$$

where all parameters  $a, b, c, d$  are positive numbers and the initial conditions  $x_0, y_0$  are nonnegative numbers. In view of the following preliminary result I will restrict our attention to the case  $a, d \in (0, 1)$ .

**Claim 2** *Consider system (55).*

1.) *If  $a \geq 1$  then  $\lim_{n \rightarrow \infty} x_n = \infty$  and  $\lim_{n \rightarrow \infty} y_n = \infty$  if  $d \geq 1$  and  $\lim_{n \rightarrow \infty} y_n = \frac{c}{1-d}$ , if  $d < 1$ .*

2.) *If  $d \geq 1$  then  $\lim_{n \rightarrow \infty} y_n = \infty$  and  $\lim_{n \rightarrow \infty} x_n = \infty$  if  $a \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = \frac{b}{1-a}$ , if  $a < 1$ .*

**Proof.**

1.) If  $a \geq 1$  then the first equation of system (55) implies  $x_{n+1} > ax_n \geq x_n$ , which shows that  $\{x_n\}_{n=1}^{\infty}$  is an increasing sequence and because there is no positive equilibrium in this case I have that  $\lim_{n \rightarrow \infty} x_n = \infty$ . In view of Theorem on difference inequalities, see [11]  $\{y_n\}_{n=1}^{\infty}$  is converging to the asymptotic solution of the limiting equation

$$y_{n+1} = c + dy_n, \quad n = 1, 2, \dots$$

which completes the proof in this case.

2.) The proof in this case is similar to the proof of case 1.) and is omitted.

□

System (55) is cooperative system with interspecific cooperation coefficients  $\frac{by_n^2}{1+y_n^2}, \frac{cx_n^2}{1+x_n^2}$  which are quadratic Beverton-Holt functions describing the interactions of two species. The simpler system

$$x_{n+1} = ax_n + \frac{by_n}{1+y_n}, \quad y_{n+1} = \frac{cx_n}{1+x_n} + dy_n, \quad n = 0, 1, \dots, \quad (56)$$

where all parameters  $a, b, c, d$  are positive numbers and the initial conditions  $x_0, y_0$  are nonnegative numbers exhibits simple exchange of stability bifurcation for the critical value of the coefficients  $bc - (1-a)(1-d)$ , see [1]. More precisely, when  $a, d \in (0, 1)$  the zero equilibrium of system (120) is globally asymptotically stable if  $bc - (1-a)(1-d) \leq 0$  and the positive equilibrium is globally asymptotically stable if  $bc - (1-a)(1-d) > 0$ . An introduction of the Beverton-Holt sigmoid function as interspecific cooperation coefficient will change the global behavior by introducing the period-two solutions which, under certain conditions could be locally stable. From modeling point of view system (55) gives an example of system which exhibits the Allee's effect along with the globally stable positive equilibrium solution relative to its basin of attraction and globally stable period-two solution relative to its basin of attraction. System (55) may have very complicated dynamics but in the cases when it has one, two or three period-two solutions I can determine its global dynamics. I am not able to find the upper bound for number of period-two solutions neither I am able to exclude the existence of the periodic solutions of other periods. One can show that system (55) does not satisfy neither  $(O+)$  nor  $(O-)$  condition which means that I can not conclude that all solutions are converging to an equilibrium solution or to a period-two solution. Thus there is a possibility that system (55) may have periodic solutions of different periods. Since system (55) is strictly cooperative the Sharkovskii's ordering holds for periodic solutions [17] and so for instance the existence of period-three solution would imply the existence of all other periodic solutions. Some monotone systems that I considered in [2, 3, 4]

all satisfy  $(O+)$  condition and so they have simpler dynamics. The examples of monotone systems that exhibit chaos are given in [82, 17]. The results that I obtain in this paper would imply that when system (55) has period-two solutions which are locally asymptotically stable, saddle points or non-hyperbolic of stable type, then every solution converges to either an equilibrium solution or to period-two solution. So the necessary condition for system (55) to have periodic solution of period 3 or higher is that all period-two solutions are either repellers or non-hyperbolic of unstable type.

The principal tool that I will use in our proofs are two results on basins of attraction of monotone maps which I provide in the rest of this section and the results on existence of stable and unstable manifolds for monotone maps in the plane from [65, 66, 67]. In the rest of this section I list basic notation which will be used in our results.

Let  $\preceq$  be a partial order on  $\mathbb{R}^n$  with nonnegative cone  $P$ . For  $x, y \in \mathbb{R}^n$  the *order interval*  $\llbracket x, y \rrbracket$  is the set of all  $z$  such that  $x \preceq z \preceq y$ . I say  $x \prec y$  if  $x \preceq y$  and  $x \neq y$ , and  $x \ll y$  if  $y - x \in \text{int}(P)$ . A map  $T$  on a subset of  $\mathbb{R}^n$  is *order preserving* if  $T(x) \preceq T(y)$  whenever  $x \prec y$ , *strictly order preserving* if  $T(x) \prec T(y)$  whenever  $x \prec y$ , and *strongly order preserving* if  $T(x) \ll T(y)$  whenever  $x \prec y$ .

Let  $T : R \rightarrow R$  be a map with a fixed point  $\bar{x}$  and let  $R'$  be an invariant subset of  $R$  that contains  $\bar{x}$ . I say that  $\bar{x}$  is *stable* (asymptotically stable) relative to  $R'$  if  $\bar{x}$  is a *stable* (asymptotically stable) fixed point of the restriction of  $T$  to  $R'$ .

Throughout this paper I shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by  $(x_1, y_1) \preceq_{ne} (x_2, y_2)$  if  $x_1 \leq x_2$  and  $y_1 \leq y_2$  and the *South-East* (SE) ordering defined as  $(x_1, y_1) \preceq_{se} (x_2, y_2)$  if  $x_1 \leq x_2$  and  $y_1 \geq y_2$ .

A map  $T$  on a nonempty set  $\mathcal{R} \subset \mathbb{R}^2$  which is monotone with respect to the

North-East (NE) ordering is called *cooperative* and a map monotone with respect to the South-East (SE) ordering is called *competitive*. A map  $T$  on a nonempty set  $\mathcal{R} \subset \mathbb{R}^2$  which second iterate  $T^2$  is monotone with respect to the North-East (resp. South-East) ordering is called *anti-cooperative* (resp. *anti-competitive*), see [7].

If  $T$  is differentiable map on a nonempty set  $\mathcal{R}$ , a sufficient condition for  $T$  to be strongly monotone with respect to the NE ordering is that the Jacobian matrix at all points  $x$  has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad (57)$$

provided that  $\mathcal{R}$  is open and convex.

For  $(x_1, x_2) \in \mathbb{R}^2$ , define  $Q_\ell(x_1, x_2)$  for  $\ell = 1, 2, 3, 4$  to be the usual four quadrants based at  $x$  and numbered in a counterclockwise direction, for example,  $Q_1(x_1, x_2) = \{y = (y_1, y_2) \in \mathbb{R}^2 : x_1 \leq y_1, x_2 \leq y_2\}$ . Basin of attraction of a fixed point  $(\bar{x}, \bar{y})$  of a map  $T$ , denoted as  $\mathcal{B}((\bar{x}, \bar{y}))$ , is defined as the set of all initial points  $(x_0, y_0)$  for which the sequence of iterates  $T^n((x_0, y_0))$  converges to  $(\bar{x}, \bar{y})$ . Similarly, I define a basin of attraction of a periodic point of period  $p$ . An ordered interval with endpoints  $a, b \in \mathbb{R}^n$  with respect to the ordering  $\preceq$  is denoted as  $\llbracket a, b \rrbracket$  and defined as  $\llbracket a, b \rrbracket = \{x \in \mathbb{R}^n : a \preceq x \preceq b\}$ .

Let  $T$  be a cooperative map defined on  $\mathcal{R} \subset \mathbb{R}^2$ . The map  $T$  is said to satisfy the property  $(O+)$  (resp.  $(O-)$ ) if

$$\text{if } x, y \in \mathcal{R} \text{ are such that } T(x) \preceq_{se} T(y), \text{ then } x \preceq_{se} y, \quad (O+)$$

resp.

$$\text{if } x, y \in \mathcal{R} \text{ are such that } T(x) \preceq_{se} T(y), \text{ then } y \preceq_{se} x. \quad (O-)$$

The well-known deMottoni-Schiaffino theorem, see [67, 82] claims that in this case for each  $x \in \mathcal{R}$ , the sequence  $\{T^n(x)\}$  (resp.  $\{T^{2n}(x)\}$ ) is eventually coordinate-

wise monotonic. Consequently, every bounded sequence  $\{T^n(x)\}$  (resp.  $\{T^{2n}(x)\}$ ) converges to a fixed point of  $T$  or to a point on the boundary of  $\mathcal{R}$ .

The next result in [67] is stated for order-preserving maps on  $\mathbb{R}^n$ . See [5] for a more general version valid in ordered Banach spaces.

**Theorem 20** *For a nonempty set  $R \subset \mathbb{R}^n$  and  $\preceq$  a partial order on  $\mathbb{R}^n$ , let  $T : R \rightarrow R$  be an order preserving map, and let  $a, b \in R$  be such that  $a \prec b$  and  $\llbracket a, b \rrbracket \subset R$ . If  $a \preceq T(a)$  and  $T(b) \preceq b$ , then  $\llbracket a, b \rrbracket$  is an invariant set and*

- i. *There exists a fixed point of  $T$  in  $\llbracket a, b \rrbracket$ .*
- ii. *If  $T$  is strongly order preserving, then there exists a fixed point in  $\llbracket a, b \rrbracket$  which is stable relative to  $\llbracket a, b \rrbracket$ .*
- iii. *If there is only one fixed point in  $\llbracket a, b \rrbracket$ , then it is a global attractor in  $\llbracket a, b \rrbracket$  and therefore asymptotically stable relative to  $\llbracket a, b \rrbracket$ .*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [5, 67], and is helpful for determining the basins of attraction of the equilibrium points.

**Corollary 4** *If the nonnegative cone of a partial ordering  $\preceq$  is a generalized quadrant in  $\mathbb{R}^n$ , and if  $T$  has no fixed points in  $\llbracket u_1, u_2 \rrbracket$  other than  $u_1$  and  $u_2$ , then the interior of  $\llbracket u_1, u_2 \rrbracket$  is either a subset of the basin of attraction of  $u_1$  or a subset of the basin of attraction of  $u_2$ .*

## 3.2 Main Results

First, I discuss the existence of the equilibrium solutions.

### 3.2.1 Equilibrium points

The equilibrium points of the system (55) satisfy the following system of equations:

$$\bar{x} = a\bar{x} + \frac{b\bar{y}^2}{1 + \bar{y}^2}, \quad \bar{y} = \frac{c\bar{x}^2}{1 + \bar{x}^2} + d\bar{y}. \quad (58)$$

It follows immediately that the zero equilibrium is a solution of (58).

Geometrically, solutions of (58) are intersections of two orthogonal rational curves:

$$\bar{x} = \frac{b\bar{y}^2}{(1-a)(1+\bar{y}^2)}, \quad \bar{y} = \frac{c\bar{x}^2}{(1-d)(1+\bar{x}^2)}. \quad (59)$$

See Figure 1. It follows from (59) that a condition for non-negative equilibrium points are  $a < 1$  and  $d < 1$ .

Rearranging (59), I have the following equations,

$$(\mathcal{E}_1) : a\bar{x} + a\bar{x}\bar{y}^2 + b\bar{y}^2 - \bar{x} - \bar{x}\bar{y}^2 = 0, \quad (\mathcal{E}_2) : c\bar{x}^2 + d\bar{y} + d\bar{x}^2\bar{y} - \bar{y} - \bar{x}\bar{y}^2 = 0. \quad (60)$$

From (60) one can see that all positive solutions of system (55) satisfy:

$$(1-a)((d-1)^2 + c^2)\bar{x}^5 - bc^2\bar{x}^4 + 2(d-1)^2(1-a)\bar{x}^3 + (1-a)(d-1)^2\bar{x} = 0. \quad (61)$$

and

$$(d-1)((1-a)^2 + b^2)\bar{y}^5 + b^2c\bar{y}^4 + 2(a-1)^2(d-1)\bar{y}^3 + (a-1)^2(d-1)\bar{y} = 0. \quad (62)$$

The left-hand side of (61) is a quintic polynomial. Since  $a < 1$  and  $d < 1$  the polynomial has coefficients which have two changes of sign. Consequently, by Descartes' rule of sign Eq.(61) has either zero, one, or two roots .

Consequently, System (55) has always the zero equilibrium and either zero, one or two positive equilibrium solutions.

These equilibrium solutions will be denoted  $E_0(0,0)$ ,  $E(\bar{x},\bar{y})$ ,  $E_{SW}(\bar{x},\bar{y})$  and  $E_{NE}(\bar{x},\bar{y})$ .

**Lemma 17** *Assume that  $a < 1$  and  $d < 1$ . Let*

$$\begin{aligned}\Delta = & 27(a-1)b^4c^8 - 32(a-1)^3c^6(d-1)^2(8(a-1)^2 + 9b^2) \\ & - 256(a-1)^3c^4(d-1)^4((a-1)^2 + b^2)\end{aligned}$$

*Consider the Equation*

$$(1-a)(c^2 + (d-1)^2)\bar{x}^4 - bc^2\bar{x}^3 + 2(1-a)(d-1)^2\bar{x}^2 + (1-a)(d-1)^2 = 0. \quad (63)$$

*Then the following holds:*

- a) All real roots of the equation (63) are positive numbers. Furthermore, if  $(\bar{x}, \bar{y})$  is real solution of the system (58) then  $\bar{x} \geq 0$  and  $\bar{y} \geq 0$ .*
- b) If  $\Delta > 0$ , then the equation (63) has zero real roots and two pairs of distinct conjugate imaginary roots. Consequently, System (55) has one equilibrium point  $E_0(0, 0)$ .*
- c) If  $\Delta < 0$ , then the equation (63) has two distinct real roots and one pair of conjugate imaginary roots. Consequently, System (55) has three equilibrium points  $E_0(0, 0)$ ,  $E_{SW}(\bar{x}, \bar{y})$  and  $E_{NE}(\bar{x}, \bar{y})$ .*
- d) If  $\Delta = 0$  then the equation (63) has one pair of conjugate imaginary roots and one real root of multiplicity two. Consequently, System (55) has two equilibrium points  $E_0(0, 0)$ , and  $E(\bar{x}, \bar{y})$ .*

**Proof.** The proof of a) follows from Descartes' Rule of Signs. Let

$$f(x) = (1-a)(c^2 + (d-1)^2)x^4 - bc^2x^3 + 2(-a-1)(d-1)^2x^2 + (1-a)(d-1)^2.$$

The following matrix, called the discrimination matrix of  $f(x)$  and  $f'(x)$  in [15], is actually the Sylvester matrix of  $f(x)$  and  $f'(x)$  with some permuted rows, is given

by

$$Discr(\tilde{f}) = \begin{pmatrix} a_4 & a_3 & a_2 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & 0 & a_0 & 0 & 0 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & 0 & a_0 & 0 \\ 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 & 0 \\ 0 & 0 & 0 & a_4 & a_3 & a_2 & 0 & a_0 \\ 0 & 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 0 \end{pmatrix},$$

where  $a_i$  is coefficient of the term  $y^i$  in the polynomial  $f(y)$ . Let  $D_k$  denote the determinant of the submatrix of  $Discr(f)$ , formed by the first  $2k$  row and the first  $2k$  columns, for  $k = 1, \dots, 4$ . So, by straightforward calculation one can see that

$$D_1 = 4(a-1)^2 (c^2 + (d-1)^2)^2$$

$$D_2 = (a-1)^2 (c^2 + (d-1)^2)^2 (3b^2c^4 - 16(a-1)^2(d-1)^2 (c^2 + (d-1)^2)),$$

$$D_3 = -4(a-1)^4 c^2 (d-1)^2 (c^2 + (d-1)^2)^2$$

$$(3b^2c^4 - c^2(d-1)^2 (16(a-1)^2 - b^2) - 16(a-1)^2(d-1)^4)$$

$$D_4 = (27(a-1)b^4c^8 - 32(a-1)^3c^6(d-1)^2 (8(a-1)^2 + 9b^2))$$

$$- 256(a-1)^3c^4(d-1)^4 ((a-1)^2 + b^2) (1-a)(a-1)^2(d-1)^4 (c^2 + (d-1)^2)^2$$

Now, I prove that if  $D_2 \geq 0$  then  $D_3 < 0$ . Indeed,  $D_2 \geq 0$  is equivalent to

$$b^2 \geq \frac{16(a-1)^2(d-1)^2 (c^2 + (d-1)^2)}{3c^4}.$$

This implies

$$b^2 > \frac{16(a-1)^2(d-1)^2 (c^2 + (d-1)^2)}{c^2 (3c^2 + (d-1)^2)}$$

which is equivalent to  $D_3 < 0$ .

Now, assume that  $\Delta > 0$ . The sign list of the sequence  $\{D_1, D_2, D_3, D_4\}$  is given by

$$[1, \text{sign}(D_2), \text{sign}(D_3), 1]. \quad (64)$$

From the previous facts it follows that the number of sign changes of the revised sign list of the list (64) is two. Now, the statement b) follows in view of Theorem 1 [15].



Assume that  $\Delta < 0$ . If  $D_2 < 0$  and  $D_3 > 0$  then I obtain that  $f(y)$  has three pairs of conjugate imaginary roots, which is a contradiction. Hence, if  $D_2 < 0$  then  $D_3 < 0$ . The sign list of the sequence  $\{D_1, D_2, D_3, D_4\}$  can have one of the following forms

$$[1, -1, -1, -1], \quad [1, 1, -1, -1], \quad [1, 1, 1, -1] \quad (65)$$

which implies that the number of sign changes of the revised sign list of (65) is one. Now, the statement c) follows in view of Theorem 1 [15]. Similarly, one can prove the statement d).  $\square$

**Lemma 18** *Assume that  $a < 1$  and  $d < 1$ . Then for  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  the rectangle*

$$\left[0, \frac{b}{1-a} + \epsilon_1\right] \times \left[0, \frac{c}{1-d} + \epsilon_2\right]$$

*is invariant and attracts all solutions of System (55).*

**Proof.** Every solution of System (55) satisfies

$$x_{n+1} \leq ax_n + b \quad y_{n+1} \leq dy_n + c.$$

The majorant system of System (55)

$$\begin{cases} u_{n+1} = au_n + b \\ v_{n+1} = dv_n + c \end{cases}$$

has a solution

$$\begin{aligned} u_n &= C_1 a^{n-1} + \frac{b(1-a^n)}{1-a} \\ v_n &= C_2 d^{n-1} + \frac{c(1-d^n)}{1-d} \end{aligned}$$

which implies that  $u_n \rightarrow \frac{b}{1-a}$  and  $v_n \rightarrow \frac{c}{1-d}$  as  $n \rightarrow \infty$ . By using the difference inequality theorem, see [11]  $x_0 = u_0$  and  $y_0 = v_0$  gives  $x_n \leq u_n$  and  $y_n \leq v_n$  which

implies  $\limsup_{n \rightarrow \infty} x_n \leq \frac{b}{1-a}$  and  $\limsup_{n \rightarrow \infty} y_n \leq \frac{c}{1-d}$ . Thus I conclude that the rectangle

$$\left[0, \frac{b}{1-a} + \epsilon_1\right] \times \left[0, \frac{c}{1-d} + \epsilon_2\right]$$

for  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  attracts all solutions. Set

$$U_1 = \frac{b}{1-a} + \epsilon_1, \quad U_2 = \frac{c}{1-d} + \epsilon_2.$$

I have that

$$\begin{aligned} T_1(x, y) &= ax + \frac{by^2}{1+y^2} \leq ax + b \leq aU_1 + b \leq U_1 \\ T_2(x, y) &= \frac{cx^2}{1+x^2} + dy \leq dy + c \leq dU_2 + b \leq U_2 \end{aligned}$$

which shows that the set  $[0, U_1] \times [0, U_2]$  is invariant.  $\square$

### 3.2.2 Local Stability of Equilibrium Solutions

Second, I discuss a local stability of the equilibrium solutions.

The equilibrium solutions of the system (55) satisfy system of equations (58). It follows immediately that the origin is a solution of (58). Geometrically, solutions of (58) are intersections of two orthogonal rational curves (59).

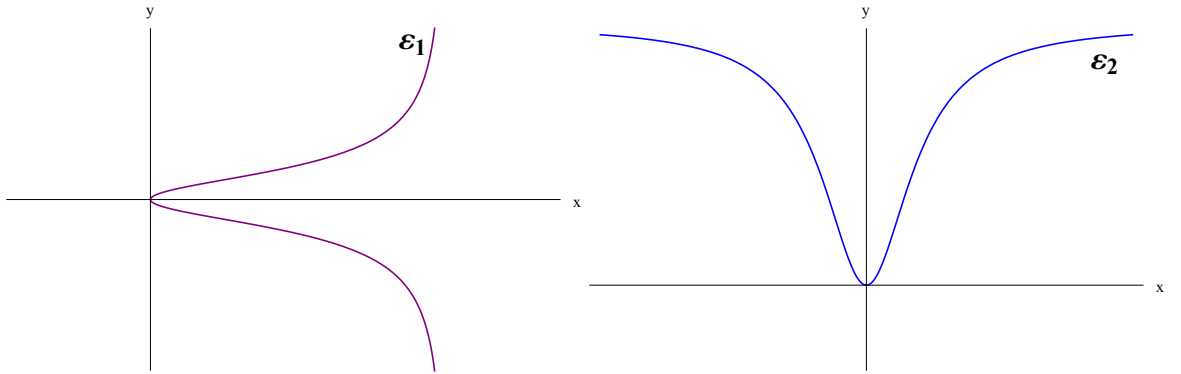


Figure 5.

figure1-1

The map associated with the system (55) has the form:

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + \frac{by^2}{1+y^2} \\ \frac{cx^2}{1+x^2} + dx \end{pmatrix}.$$

The Jacobian matrix of  $T$  is

$$J_T(x, y) = \begin{pmatrix} a & \frac{2by}{(1+y^2)^2} \\ \frac{2cx}{(1+x^2)^2} & d \end{pmatrix}. \quad (66)$$

The Jacobian matrix of  $T$  evaluated in an equilibrium point with positive coordinates  $(\bar{x}, \bar{y})$  has the form:

$$J_T(\bar{x}, \bar{y}) = \begin{pmatrix} a & \frac{2\bar{x}(1-a)}{\bar{y}(1+\bar{y}^2)} \\ \frac{2\bar{y}(1-d)}{\bar{x}(1+\bar{x}^2)} & d \end{pmatrix}. \quad (67)$$

The determinant and trace of (67) are:

$$\det(J_T(\bar{x}, \bar{y})) = ad - \frac{2(1-d)}{(1+\bar{x}^2)} \frac{2(1-a)}{(1+\bar{y}^2)}, \quad \text{tr}(J_T(\bar{x}, \bar{y})) = a + d. \quad (68)$$

The eigenvalues of (67) are

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{(1+\bar{x}^2)} \frac{2(1-a)}{(1+\bar{y}^2)}}}{2}, \quad \mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{(1+\bar{x}^2)} \frac{2(1-a)}{(1+\bar{y}^2)}}}{2} \quad (69)$$

with corresponding eigenvectors

$$E_\lambda = \begin{pmatrix} \frac{\bar{x}(1+\bar{x}^2)}{2(1-d)\bar{y}} \left( \frac{(a-d) + \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{(1+\bar{x}^2)} \frac{2(1-a)}{(1+\bar{y}^2)}}}{2} \right), 1 \end{pmatrix}, \quad (70)$$

$$E_\mu = \begin{pmatrix} \frac{\bar{x}(1+\bar{x}^2)}{2(1-d)\bar{y}} \left( \frac{(a-d) - \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{(1+\bar{x}^2)} \frac{2(1-a)}{(1+\bar{y}^2)}}}{2} \right), 1 \end{pmatrix}$$

It is clear from (69) that  $\lambda$  and  $\mu$  are real numbers such that  $\lambda > \mu$  and  $\lambda > 0$ .

**Lemma 19** *The following conditions hold for the coordinates of the positive equilibrium points of System (55).*

(i) For  $E_{NE}(\bar{x}, \bar{y})$

$$(\bar{x}^2 + 1)(\bar{y}^2 + 1) > 4; \quad (71)$$

(ii) For  $E_{SW}(\bar{x}, \bar{y})$

$$(\bar{x}^2 + 1)(\bar{y}^2 + 1) < 4; \quad (72)$$

(ii) For  $E(\bar{x}, \bar{y})$

$$(\bar{x}^2 + 1)(\bar{y}^2 + 1) = 4. \quad (73)$$

**Proof.**

(i) Let  $m_{\mathcal{E}_1}$  be the slope of the tangent line to rational equation  $\mathcal{E}_1$  at  $E_{NE}(\bar{x}, \bar{y})$  and let  $m_{\mathcal{E}_2}$  be the slope of the tangent line to rational equation  $\mathcal{E}_2$  at  $E_{NE}(\bar{x}, \bar{y})$ . It is clear from geometry that

$$m_{\mathcal{E}_1} > m_{\mathcal{E}_2}.$$

See Figure 2. It follows that

$$\frac{dy}{dx}|_{\mathcal{E}_1}(\bar{x}, \bar{y}) > \frac{dy}{dx}|_{\mathcal{E}_2}(\bar{x}, \bar{y})$$

and in turn

$$\frac{(\bar{y}^2 + 1)^2(1 - a)}{2b\bar{y}} > \frac{2c\bar{x}}{(\bar{x}^2 + 1)^2(1 - d)}$$

which is equivalent to

$$(1 - a)(1 - d) > \frac{2c\bar{x}2b\bar{y}}{(\bar{x}^2 + 1)^2(\bar{y}^2 + 1)^2}.$$

Using the equilibrium condition (58) I may rewrite this

$$(1-a)(1-d) > \frac{2(1-a)2(1-d)}{(\bar{x}^2+1)(\bar{y}^2+1)}$$

The proofs for cases (ii) and (iii) are similar and will be omitted.

□

**Theorem 21**  $E_0(0,0)$  is locally asymptotically stable.

**Proof.** The eigenvalues of (66) evaluated at  $E_0(0,0)$  are  $\lambda = a$  and  $\mu = b$ , where  $0 < a < 1$  and  $0 < b < 1$ . □

**Theorem 22** When System (55) has one positive equilibrium point,  $E(\bar{x}, \bar{y})$  is non-hyperbolic of the stable type.

**Proof.** I need to show that  $\lambda = 1$  and  $-1 < \mu < 1$ . I will first show that  $\lambda = 1$ . From (73) and the fact that  $a < 1$  and  $d < 1$ , I have

$$\begin{aligned} \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}} &= \sqrt{(a-d)^2 + 4(1-a)(1-d)} \\ &= \sqrt{(a+d-2)^2} = |a+d-2| = 2-d-a. \end{aligned}$$

Therefore

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}}}{2} = 1.$$

I will next show  $-1 < \mu < 1$ . Since by (69) it is clear that  $\mu < \lambda$ , and I have shown that  $\lambda = 1$ , it follows that  $\mu < 1$ . From  $a < 1$ ,  $b < 1$  and

$$\sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}} = 2-d-a$$

it follows that

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} < 2 + d + a.$$

Therefore

$$-1 < \frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2} = \mu.$$

□

**Theorem 23** *When System (55) has two positive equilibrium points,  $E_{NE}(\bar{x}, \bar{y})$  is locally asymptotically stable.*

**Proof.** I will first show that  $0 < \lambda < 1$ . Indeed

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2} > 0.$$

From (71) and the fact that  $a < 1$  and  $d < 1$ , I have

$$\begin{aligned} \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} &< \sqrt{(a-d)^2 + 4(1-a)(1-d)} \\ &= \sqrt{(a+d-2)^2} = |a+d-2| = 2-d-a. \end{aligned}$$

Therefore

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} < 2-d-a$$

and

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2} < 1.$$

I will next show that  $-1 < \mu < 1$ .

From (71) and the fact that  $a < 1$  and  $d < 1$ , I have

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} < \sqrt{(a-d)^2 + 4(1-a)(1-d)}$$

$$= \sqrt{(a+d-2)^2} = |a+d-2| = 2-d-a < 2+a+d.$$

Therefore

$$\sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} < 2+a+d,$$

and

$$-2 < (a+d) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}$$

which yields

$$-1 < \frac{(a+d) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2}$$

and so

$$-1 < \mu.$$

Since  $a < 1$  and  $d < 1$ , I have

$$d+a-2 < 0 < \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}.$$

Therefore

$$(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}} < 2$$

and

$$\frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2} < 1,$$

that is  $\mu < 1$ . □

**Theorem 24** *When System (55) has two positive equilibrium points,  $E_{SW}(\bar{x}, \bar{y})$  is either a repeller, nonhyperbolic of the unstable type, or a saddle point.*

**Proof.** I need to show that  $\lambda > 1$  and either  $\mu < -1$ ,  $\mu = -1$ , or  $-1 < \mu < 1$ . I will first show that  $\lambda > 1$ . From (72) and the fact that  $a < 1$  and  $d < 1$ , I have

$$\begin{aligned} \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}} &> \sqrt{(a-d)^2 + 4(1-a)(1-d)} \\ &= \sqrt{(a+d-2)^2} = |a+d-2| = 2-d-a. \end{aligned}$$

Therefore

$$\sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}} > 2-d-a$$

and

$$\lambda = \frac{(d+a) + \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}}}{2} > 1.$$

I will next show that  $\mu < 1$ . Suppose that  $\mu \geq 1$ . Then

$$\mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}}}{2} \geq 1$$

which is equivalent to

$$d+a-2 \geq \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}}.$$

This is a contradiction since  $a < 1$  and  $d < 1$  imply that

$$d+a-2 < 0 < \sqrt{(a-d)^2 + 4 \frac{2(1-d)}{1+x^2} \frac{2(1-a)}{1+y^2}}.$$

Therefore  $\mu < 1$ .



I will next show that either  $\mu < -1$ ,  $\mu = -1$ , or  $1 > \mu > -1$ . Since this covers the remaining parametric space, it is sufficient to show that there exist real values of  $a, b, c$ , and  $d$  for which each aforementioned value of  $\mu$  exists.

(i) Case  $\mu < -1$

$$\text{When } a = \frac{8}{100}, b = \frac{841}{395}, c = \frac{841}{395}, d = \frac{8}{100}, \mu = -\frac{69\sqrt{497341}}{105125} - \frac{21}{25} = -1.30289.$$

(ii) Case  $\mu = -1$

$$\text{When } a = \frac{21}{79}, b = \frac{841}{395}, c = \frac{841}{395}, d = \frac{21}{79}, \mu = -1.$$

(iii) Case  $-1 < \mu < 1$

$$\text{When } a = \frac{3}{10}, b = \frac{841}{395}, c = \frac{841}{395}, d = \frac{3}{10}, \mu = -\frac{84\sqrt{697}}{4205} - \frac{2}{5} = -.927.$$

□

**Theorem 25** *When System (55) has two positive equilibrium points, the following conditions hold for  $E_{SW}(\bar{x}, \bar{y})$ .*

(i)  $E_{SW}(\bar{x}, \bar{y})$  is a repeller when

$$\bar{x}\bar{y} < \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

(ii)  $E_{SW}(\bar{x}, \bar{y})$  is nonhyperbolic of the unstable type when

$$\bar{x}\bar{y} = \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

(iii)  $E_{SW}(\bar{x}, \bar{y})$  is a saddle point when

$$\bar{x}\bar{y} > \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}.$$

**Proof.**

(i) By (59), when  $xy < \frac{4(1-a)^2(1-d)^2}{bc(1+a)(1+d)}$ ,

$$(1+a)(1+d) < \frac{4(1-a)(1-d)}{(1+x^2)(1+y^2)}.$$

It follows that

$$\mu = \frac{(d+a) - \sqrt{(a-d)^2 + 4\frac{2(1-d)}{1+x^2}\frac{2(1-a)}{1+y^2}}}{2} < -1$$

The proofs of (ii) and (iii) are similar and will be omitted.

□

### 3.2.3 Global behavior

**The case  $\Delta < 0$**

Assume that  $\Delta < 0$ . By Lemma 17, system (55) has three equilibrium solutions  $E_0(0,0)$ ,  $E_{SW}(\bar{x}, \bar{y})$  and  $E_{NE}(\bar{x}, \bar{y})$ . By Theorems 21 and 23 the equilibrium points  $E_0(0,0)$  and  $E_{NE}(\bar{x}, \bar{y})$  are locally asymptotically stable. One can see that  $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$ .

Let  $\mathcal{B}(E_0)$  be the basin of attraction of  $E_0(0,0)$  and  $\mathcal{B}(E_{NE})$  be the basin of attraction of  $E_{NE}$ .

The following lemma holds.

**Lemma 20** *Let  $E_{SW}(\bar{x}_{SW}, \bar{y}_{SW})$ . The following hold:*

(i) *If  $Q_1(E_{SW}) = \{(x, y) : x \geq \bar{x}_{SW} \text{ and } y \geq \bar{y}_{SW}\}$  then  $\text{int}(Q_1(E_{SW})) \subset \mathcal{B}(E_{NE})$ .*

(ii) *If  $Q_3(E_{SW}) = \{(x, y) : 0 \leq x \leq \bar{x}_{SW} \text{ and } 0 \leq y \leq \bar{y}_{SW}\}$  then  $\text{int}(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$ .*

**Proof.** By Corollary 4 I obtain  $\text{int}(Q_3(E_{SW})) \subset \mathcal{B}(E_0)$  and  $\text{int}(Q_1(E_{SW}) \cap Q_3(E_{NE})) \subset \mathcal{B}(E_{NE})$ . In view of Lemma 18 I have that for  $(x_0, y_0) \in \text{int}(Q_1(E_{NE}))$  there exists  $n_0$  such that  $T^n(x_0, y_0) \in \llbracket E_{NE}, (U_1, U_2) \rrbracket_{ne}$  for all  $n > n_0$ . Since  $T$  is cooperative map,  $T(\llbracket E_{NE}, (U_1, U_2) \rrbracket_{ne}) \subseteq \llbracket E_{NE}, (U_1, U_2) \rrbracket_{ne}$  and  $E_{NE}$  is only equilibrium, I obtain that  $\llbracket E_{NE}, (U_1, U_2) \rrbracket_{ne} \subset \mathcal{B}(E_{NE})$ . Consequently, I obtain that  $\text{int}(Q_1(E_{NE})) \subset \mathcal{B}(E_{NE})$ . Assume that  $(x_0, y_0) \in \text{int}(Q_1(E_{SW}))$ . Then, there exists  $(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E_{SW}) \cap Q_3(E_{NE}))$  such that  $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0)$  and  $(\tilde{x}_1, \tilde{y}_1) \in \text{int}(Q_1(E_{NE}))$  such that  $(x_0, y_0) \preceq_{ne} (\tilde{x}_1, \tilde{y}_1)$ . By monotonicity of  $T$  I have  $T^n(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(\tilde{x}_1, \tilde{y}_1)$  which implies  $T^n(x_0, y_0) \rightarrow E_{NE}$  as  $n \rightarrow \infty$ . This implies that  $\text{int}(Q_1(E_{SW})) \subset \mathcal{B}(E_{NE})$ . The proof of (ii) follows from Corollary 4 applied to the ordered interval  $\llbracket E_0, E_{SW} \rrbracket$ .  $\square$

Let  $\mathcal{C}_1^+$  denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_2(E_{SW})$  and  $\mathcal{C}_1^-$  denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_4(E_{SW})$ . Also, let  $\mathcal{C}_2^+$  denote the boundary of  $\mathcal{B}(E_{NE})$  considered as a subset of  $Q_2(E_{SW})$  and  $\mathcal{C}_2^-$  denote the boundary of  $\mathcal{B}(E_0)$  considered as a subset of  $Q_4(E_{SW})$ . It is easy to see that  $E_{SW} \in \mathcal{C}_1^+$ ,  $E_{SW} \in \mathcal{C}_1^-$ ,  $E_{SW} \in \mathcal{C}_2^+$ ,  $E_{SW} \in \mathcal{C}_2^-$ , and  $T(\mathcal{R}) \subset \text{int}(\mathcal{R})$ .

The proof of the following Lemmas for cooperative map is the same as the proof of Claims 1 and 2 [4] for competitive map, so I skip it (See Figure 3).

**Lemma 21** *Let  $\mathcal{C}_1^+$  and  $\mathcal{C}_1^-$  be the sets defined above. Then*

- a) *If  $(x_0, y_0) \in \mathcal{B}(E_0)$  then  $(x_1, y_1) \in \mathcal{B}(E_0)$  for all  $(x_1, y_1) \preceq_{ne} (x_0, y_0)$ .*
- b) *If  $(x_0, y_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$  then  $(x_1, y_1) \in \text{int}(\mathcal{B}(E_0))$  for all  $(x_1, y_1) \ll_{ne} (x_0, y_0)$ .*
- c)  *$\mathcal{C}_1^+ \cap \text{int}(Q_2(E_{SW})) \neq \emptyset$  and  $\mathcal{C}_1^- \cap \text{int}(Q_4(E_{SW})) \neq \emptyset$ .*
- d)  *$T(\mathcal{C}_1^+ \cup \mathcal{C}_1^-) \subseteq \mathcal{C}_1^+ \cup \mathcal{C}_1^-$ .*
- e)  *$(x_0, y_0), (x_1, y_1) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^- \Rightarrow (x_0, y_0) \ll_{se} (x_1, y_1)$  or  $(x_1, y_1) \ll_{se} (x_0, y_0)$ .*

f)  $\mathcal{C}_1^+ \cup \mathcal{C}_1^-$  is the graph of continuous strictly decreasing function.

**Lemma 22** Let  $\mathcal{C}_2^+$  and  $\mathcal{C}_2^-$  be the sets defined as above. Then

- a) If  $(x_0, y_0) \in \mathcal{B}(E_{NE})$  then  $(x_1, y_1) \in \mathcal{B}(E_{NE})$  for all  $(x_0, y_0) \preceq_{ne} (x_1, y_1)$ .
- b) If  $(x_0, y_0) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$  then  $(x_1, y_1) \in \text{int}(\mathcal{B}(E_{NE}))$  for all  $(x_0, y_0) \ll_{ne} (x_1, y_1)$ .
- c)  $\mathcal{C}_2^+ \cap \text{int}(Q_2(E_{SW})) \neq \emptyset$  and  $\mathcal{C}_2^- \cap \text{int}(Q_4(E_{SW})) \neq \emptyset$ .
- d)  $T(\mathcal{C}_2^+ \cup \mathcal{C}_2^-) \subseteq \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .
- e)  $(x_0, y_0), (x_1, y_1) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^- \Rightarrow (x_0, y_0) \ll_{se} (x_1, y_1)$  or  $(x_1, y_1) \ll_{se} (x_0, y_0)$ .
- f)  $\mathcal{C}_2^+ \cup \mathcal{C}_2^-$  is the graph of continuous strictly decreasing function.

By Lemmas 22 and 21 it remains to determine the behavior of the orbits of initial conditions  $(x_0, y_0)$  such that  $(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (x_0, y_0) \preceq (\bar{x}_0, \bar{y}_0)$  for some  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{C}_1^+ \cup \mathcal{C}_1^-$  and  $(\bar{x}_0, \bar{y}_0) \in \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .

Now, I present global dynamics of system (55) in different parametric regions, based on our numerical simulations. The cases I consider depend on existence or non-existence of period-two solutions.

**Theorem 26** If  $\Delta < 0$  and a minimal period-two solution does not exist, then System (55) has three equilibrium points  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is unstable. If  $E_{SW}$  is a saddle point then there exist two continuous curves  $\mathcal{W}^s(E_{SW})$  and  $\mathcal{W}^u(E_{SW})$ , both passing through the point  $E_{SW}$ , such that  $\mathcal{W}^s(E_{SW})$  is a graph of decreasing function and  $\mathcal{W}^u(E_{SW})$  is a graph of an increasing function. The first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of three disjoint basins of attraction, namely  $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$ , where  $\mathcal{B}(E_{SW}) = \mathcal{W}^s(E_{SW})$  and

$$\begin{aligned} \mathcal{B}(E_0) &= \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\} \\ \mathcal{B}(E_{NE}) &= \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}. \end{aligned}$$

Thus, I have  $\mathcal{W}^s(E_{SW}) = \mathcal{C}_1^+ \cup \mathcal{C}_1^- = \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .

**Proof.** Lemma 17 implies that there exist three equilibrium point namely  $E_0$ ,  $E_{SW}$  and  $E_{NE}$  such that  $E_0 \ll_{ne} E_{SW} \ll_{ne} E_{NE}$ . In this case,  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is a saddle point. In view of (66) the map  $T$  is strongly cooperative on  $[0, \infty)^2$ . It follows from the Perron-Frobenius Theorem and a change of variables [82] that at each point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly cooperative then no eigenvector is aligned with a coordinate axis.

Hence, all conditions of Theorems 1 and 4 from [66] for cooperative map  $T$  are satisfied, which yields the existence of the global stable manifold  $\mathcal{W}^s(E_{SW})$  and the global unstable manifold  $\mathcal{W}^u(E_{SW})$ , where  $\mathcal{W}^s(E_{SW})$  is passing through the point  $E_{SW}$ , and it is a graph of decreasing function. The curve  $\mathcal{W}^u(E_{SW})$  is the graph of an increasing function. Let

$$\mathcal{W}^- = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(E_{SW})\},$$

$$\mathcal{W}^+ = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(E_{SW})\}.$$

Take  $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$  and  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$ . By Theorem 4 [66] I have that there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(E_{SW}))$  and  $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E_{SW}))$  for  $n > n_0$ . The rest of the proof follows from Lemma 20. See Figure 4 a) for its visual illustration.  $\square$

If  $\Delta < 0$  and  $E_{SW}$  is non-hyperbolic equilibrium point, then by Theorem 24 I have that  $\lambda > 1$  and  $\mu = -1$ . Based on a series of numerical simulations, I propose the following conjecture. See Figure 4 b) for its visual illustration.

**Conjecture 1** *If  $\Delta < 0$  and the minimal period-two solution does not exist, then System (55) has three equilibrium points  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is unstable. If  $E_{SW}$  is non-hyperbolic equilibrium point then there exists the continuous curve  $\mathcal{C}(E_{SW})$  passing through the point  $E_{SW}$ , such that  $\mathcal{C}(E_{SW})$  is a graph of decreasing function. The first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of three disjoint basins of attraction, namely  $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_{SW}) \cup \mathcal{B}(E_{NE})$ , where  $\mathcal{B}(E_{SW}) = \mathcal{W}^s(E_{SW})$  and*

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(E_{SW})\}$$

$$\mathcal{B}(E_{NE}) = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(E_{SW})\}.$$

*Thus, I have  $\mathcal{C}(E_{SW}) = \mathcal{C}_1^+ \cup \mathcal{C}_1^- = \mathcal{C}_2^+ \cup \mathcal{C}_2^-$ .*

Now, I consider the dynamical scenarios when there exists a minimal period-two solution which is a saddle point. For numerical values, see Figure 5 a).

**Theorem 27** *Assume that  $\Delta < 0$  and there exists a minimal period-two solution  $\{P_1, T(P_1)\}$  which is a saddle point. Then System (55) has three equilibrium solutions  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is a repeller or non-hyperbolic point. In this case there exist four continuous curves  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^u(P_1), \mathcal{W}^u(T(P_1))$ , where  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1))$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. The curves  $\mathcal{W}^u(P_1), \mathcal{W}^u(T(P_1))$  are the graphs of increasing functions and are starting at  $E_0$ . Every solution which starts below  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  in the North-east ordering converges to  $E_0$  and every solution which starts above  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  in the North-east ordering converges to  $E_{NE}$ , i.e.  $\mathcal{W}^s(P_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$  and  $\mathcal{W}^s(T(P_1)) = \mathcal{C}_1^- = \mathcal{C}_2^-$ .*

**Proof.** Since a square  $T^2$  of cooperative map  $T$  is cooperative, all conditions of Theorems 1. and 4. from [66] are satisfied, which yields the existence of

the global stable manifolds  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1))$  and the global unstable manifolds  $\mathcal{W}^u(P_1), \mathcal{W}^u(T(P_1))$  where  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1))$  are passing through the point  $E_{SW}$ , and they are graphs of decreasing functions. Let

$$\begin{aligned}\mathcal{W}^- &= \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}, \\ \mathcal{W}^+ &= \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \\ &\quad \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}.\end{aligned}$$

Take  $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$  and  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$ . By Theorem 4. [66] I have that there exist  $n_0, n_1 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$  for  $n > n_0$  and  $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(P_1) \cap Q_1(T(P_1)))$  for  $n > n_1$ . So, it is enough to prove that  $\text{int}(Q_3(P_1) \cap Q_3(T(P_1))) \subseteq \mathcal{B}(E_0)$  and  $\text{int}(Q_1(P_1) \cap Q_1(T(P_1))) \subseteq \mathcal{B}(E_{NE})$ . Indeed, by Theorem 6 [66] for any  $(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$  there exists subsolution  $(\tilde{x}_0, \tilde{y}_0)$  (i.e.  $T(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} (\tilde{x}_0, \tilde{y}_0)$ ) such that  $(x_0, y_0) \preceq_{ne} (\tilde{x}_0, \tilde{y}_0)$ . Since  $E_0 \preceq_{ne} T^{2n+2}(\tilde{x}_0, \tilde{y}_0) \preceq_{ne} T^{2n}(\tilde{x}_0, \tilde{y}_0)$  and there is only one period-two solution in  $\text{int}(Q_3(P_1) \cap Q_3(T(P_1)))$  I obtain  $T^{2n}(\tilde{x}_0, \tilde{y}_0) \rightarrow E_0$  as  $n \rightarrow \infty$ . From  $T^{2n}(x_0, y_0) \preceq_{ne} T^{2n}(\tilde{x}_0, \tilde{y}_0)$  I have that  $T^{2n}(x_0, y_0) \rightarrow E_0$  as  $n \rightarrow \infty$ . Since  $T$  is continuous function in the first quadrant I have  $T^n(x_0, y_0) \rightarrow E_0$  as  $n \rightarrow \infty$ . Similarly, I have that  $T^n(x_0, y_0) \rightarrow E_{NE}$  as  $n \rightarrow \infty$  if  $(x_0, y_0) \in \text{int}(Q_1(P_1) \cap Q_1(T(P_1)))$ . This complete the proof.  $\square$

Now, I consider the dynamical scenario when there exist three minimal period-two solutions. For numerical values, see Figure 5 b).

**Theorem 28** *Assume that  $\Delta < 0$  and there exist three minimal period-two solutions  $\{P_i, T(P_i)\}$   $i = 1, 2, 3$ , where  $\{P_1, T(P_1)\}$  and  $\{P_2, T(P_2)\}$  are the saddle points and  $\{P_3, T(P_3)\}$  is locally asymptotically stable. Then System (55) has three equilibrium solutions  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is repeller or non-hyperbolic equilibrium point. In this case there exist four continuous curves  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2))$  where*

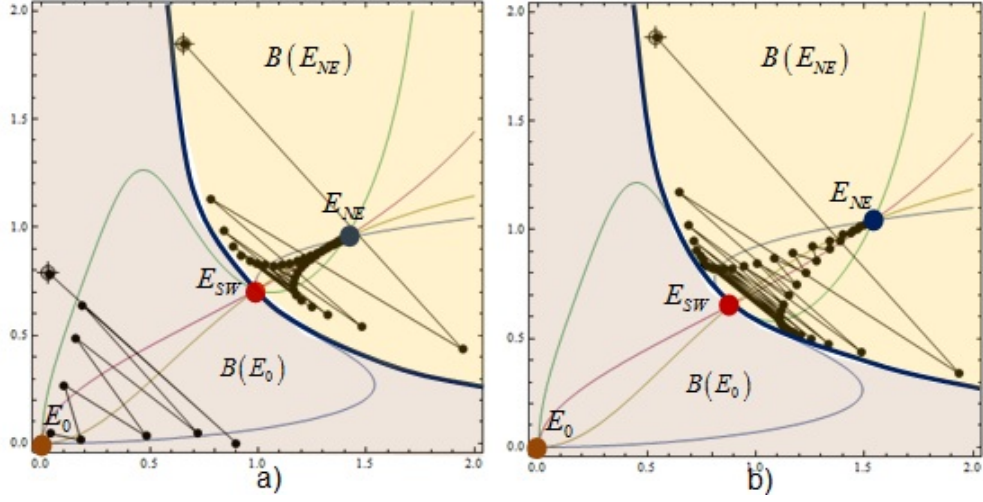


Figure 6. a) Visual illustration of Theorem 26 when  $a = 0.21$ ,  $b = 2.34$ ,  $c = 1.43$  and  $d = 0.01$ , in the case when  $E_{SW}$  is a saddle point. b) Visual illustration of Conjecture 1 when  $a = 0.20738$ ,  $b = 2.34$ ,  $c = 1.47$  and  $d = 0.01$ , in the case when  $E_{SW}$  is non-hyperbolic equilibrium.

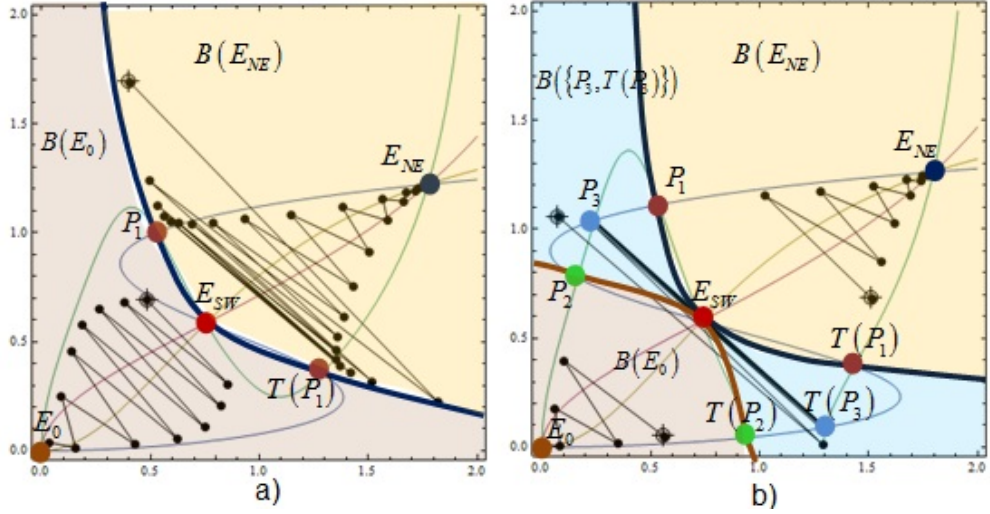


Figure 7. a) Visual illustration of Theorem 27 when  $a = 0.21$ ,  $b = 2.34$ ,  $c = 1.61$  and  $d = 0.02$ . The case when  $E_{SW}$  is a repeller and  $\{P_1, T(P_1)\}$  is a period two-solution which is a saddle point. b) Visual illustration of Theorem 28 when  $a = 0.17$ ,  $b = 2.43$ ,  $c = 1.65$  and  $d = 0.01$ . The case when  $E_{SW}$  is a repeller,  $\{P_1, T(P_1)\}$  and  $\{P_2, T(P_2)\}$  are period two-solutions which are saddle points and  $\{P_3, T(P_3)\}$  is the period two-solution which is locally asymptotically stable.



$\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2))$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. Every solution which starts below  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  in the North-east ordering converges to  $E_0$  and every solution which starts above  $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$  in the North-east ordering converges to  $E_{NE}$ . Every solution which starts above  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  and below  $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$  in the North-east ordering converges to  $\{P_3, T(P_3)\}$ . In other words, the first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of five disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1, T(P_1)\}) \cup \mathcal{B}(\{P_2, T(P_2)\}) \cup \mathcal{B}(\{P_3, T(P_3)\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_1, T(P_1)\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1)),$$

$$\mathcal{B}(\{P_2, T(P_2)\}) = \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2)),$$

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\},$$

$$\mathcal{B}(E_{NE}) = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\},$$

$$\mathcal{B}(\{P_3, T(P_3)\}) = \{(x, y) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x, y) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}.$$

Thus, I have

$$\mathcal{W}^s(P_2) = \mathcal{C}_1^+, \mathcal{W}^s(T(P_2)) = \mathcal{C}_1^-, \mathcal{W}^s(P_1) = \mathcal{C}_2^+, \text{ and } \mathcal{W}^s(T(P_1)) = \mathcal{C}_2^-.$$

**Proof.** All conditions of Theorems 1. and 4. in [66] for cooperative map  $T^2$  are satisfied, which yields the existence of the global stable manifolds  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2))$  which are passing through the point  $E_{SW}$ ,

and they are graphs of decreasing functions. Since  $T$  is cooperative map it is easy to see that  $P_1 \ll_{ne} P_3 \ll_{ne} P_2$  or  $P_2 \ll_{ne} P_3 \ll_{ne} P_1$ . Assume  $P_2 \ll_{ne} P_3 \ll_{ne} P_1$ . As in the proof of Theorem 27 one can see that

$$\begin{aligned}\mathcal{B}(E_0) &= \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \\ &\quad \text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}, \\ \mathcal{B}(E_{NE}) &= \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \\ &\quad \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\}.\end{aligned}$$

So, I assume that  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1)$ . By Theorem 4. in [66] I have that there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in \text{int}(Q_3(P_1) \cap Q_1(P_2))$  for  $n > n_0$ . By Corollary 4 I get  $[[P_2, P_3]] \cup [[P_3, P_1]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$  which implies that  $\text{int}(Q_3(P_1) \cap Q_1(P_2)) = [[P_2, P_1]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$ . If  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(T(P_2))$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{W}^s(T(P_1))$  then there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in \text{int}(Q_3(T(P_1)) \cap Q_1(T(P_2)))$  for  $n > n_0$ . By Corollary 4 I get  $[[T(P_2), T(P_3)]] \cup [[T(P_3), T(P_1)]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$  which implies that  $\text{int}(Q_3(T(P_1)) \cap Q_1(T(P_2))) = [[T(P_2), T(P_1)]] \subseteq \mathcal{B}(\{P_3, T(P_3)\})$ . This completes the proof.  $\square$

Now, I consider two dynamical scenarios when there exists a minimal period-two solution  $\{P, T(P)\}$  which is a non-hyperbolic of stable type (i.e. if  $\mu_1$  and  $\mu_2$  are eigenvalues of  $J_{T^2}(P)$  then  $\mu_1 = 1$  and  $|\mu_2| < 1$ ).

**Theorem 29** *Assume that  $\Delta < 0$  and there exist two minimal period-two solutions  $\{P, T(P)\}$  and  $\{P_1, T(P_1)\}$ , where  $\{P, T(P)\}$  is a non-hyperbolic period-two solution of the stable type and  $\{P_1, T(P_1)\}$  is a saddle point, and  $P \ll_{ne} P_1$  (See Figure 6 a). Then System (55) has three equilibrium points  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is*

repeller or non-hyperbolic equilibrium point. In this case there exist four continuous curves  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{C}^s(P), \mathcal{C}^s(T(P))$  where  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{C}(P), \mathcal{C}(T(P))$  are passing through the point  $E_{SW}$ , which are graphs of decreasing functions. The first quadrant of initial conditions  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of four disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1, T(P_1)\}) \cup \mathcal{B}(\{P, T(P)\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_1, T(P_1)\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1)),$$

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\},$$

$$\mathcal{B}(E_{NE}) = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\},$$

$$\mathcal{B}(\{P, T(P)\}) = \mathcal{C}(P) \cup \mathcal{C}(T(P)) \cup \{(x, y) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x, y) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\}.$$

Thus, I have

$$\mathcal{C}(P) = \mathcal{C}_1^+, \mathcal{C}(T(P)) = \mathcal{C}_1^-, \mathcal{W}^s(P_1) = \mathcal{C}_2^+, \text{ and } \mathcal{W}^s(T(P_1)) = \mathcal{C}_2^-.$$

**Proof.** Since  $\{P, T(P)\}$  is a non-hyperbolic period-two solution of the stable type and  $\{P_1, T(P_1)\}$  is a saddle point, all conditions of Theorems 1 and 4 [66] for cooperative map  $T^2$  are satisfied, which yields the existence of the global stable manifolds  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1))$  and invariant curves  $\mathcal{C}(P), \mathcal{C}(T(P))$  which are passing through the point  $E_{SW}$ , and they are graphs of decreasing functions.

Take  $(x_0, y_0)$  such that  $(x_0, y_0) \preceq_{ne} (x_{E_0}, y_{E_0})$  for some  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\}$ . In view of Theorem 4 [66] there exists  $n_0 > 0$  such that,  $T^{2n_0}(x_0, y_0) \in$

$int(Q_3(P) \cap Q_3(T(P)))$  for  $n > n_0$ . Since  $int(Q_3(P) \cap Q_3(T(P))) \subseteq int(Q_3(E_{SW}))$  by Lemma 20 I obtain

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\}.$$

If  $(x_0, y_0)$  such that  $(x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x_0, y_0)$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  by Theorem 4 [66] there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in int(Q_1(P_1) \cap Q_1(T(P_1)))$  for  $n > n_0$ . Since  $int(Q_1(P_1) \cap Q_1(T(P_1))) \subseteq int(Q_1(E_{SW}))$  by Lemma 20 I obtain

$$\begin{aligned} \mathcal{B}(E_{NE}) = & \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y) \\ & \text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}. \end{aligned}$$

Now, I assume that  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_1)$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(P)$ . By Theorem 4 [66] I have that there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in int(Q_3(P_1) \cap Q_1(P))$  for  $n > n_0$ . By Corollary 4 I get  $[[P, P_1]] \subseteq \mathcal{B}(\{P, T(P)\})$ . Similarly, if  $(x_{E_0}, y_{E_0}) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}})$  for some  $(x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(T(P_1))$  and  $(x_{E_0}, y_{E_0}) \in \mathcal{C}(T(P))$  I have that there exists  $n_0 > 0$  such that,  $T^{2n}(x_0, y_0) \in int(Q_3(T(P_1)) \cap Q_1(T(P)))$  for  $n > n_0$ . By Corollary 4 I get  $[[T(P), T(P_1)]] \subseteq \mathcal{B}(\{P, T(P)\})$ . This implies  $(x_0, y_0) \in \mathcal{B}(\{P, T(P)\})$ .  $\square$

The proof of the following Theorem is similar to the proof of Theorem 29 and will be omitted.

**Theorem 30** *Assume that  $\Delta < 0$  and there exist two minimal period-two solutions  $\{P, T(P)\}$  and  $\{P_1, T(P_1)\}$ , where  $\{P, T(P)\}$  is a non-hyperbolic equilibrium solution of stable type and  $\{P_1, T(P_1)\}$  is a saddle point, and  $P_2 \ll_{ne} P$  (See Figure 6 b)). Then System (55) has three equilibrium solutions  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is repeller or non-hyperbolic equilibrium point. In this case there exist four continuous curves  $\mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2)), \mathcal{C}^s(P), \mathcal{C}^s(T(P))$  where  $\mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2)),$*

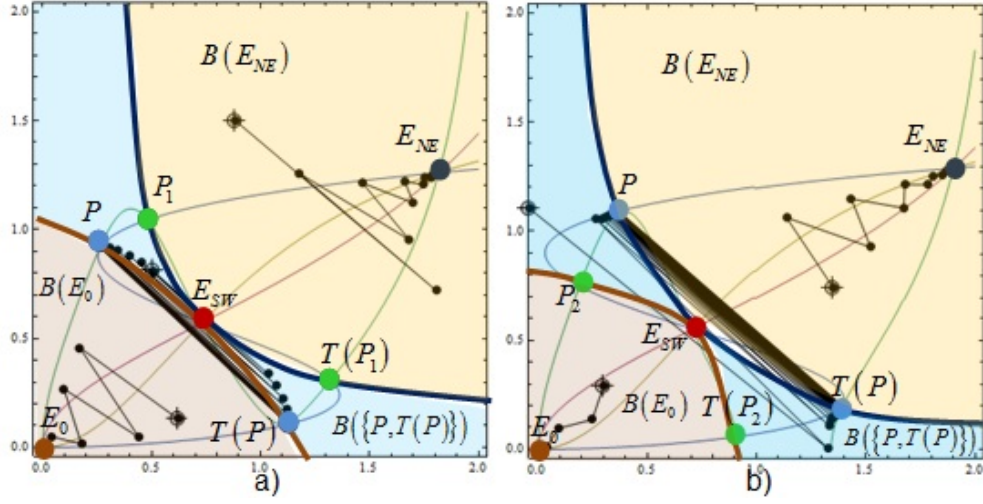


Figure 8. a) Visual illustration of Theorem 29 when  $a = 0.210703$ ,  $b = 2.34$ ,  $c = 1.638$  and  $d = 0.01$ . The case when  $E_{SW}$  is a repeller,  $\{P, T(P)\}$  is the period two-solution which is non-hyperbolic ( $\mu_1 = 1$  and  $\mu_2 = 0.3797$ ) and  $\{P_1, T(P_1)\}$  is the period two-solution which is a saddle point. b) Visual illustration of Theorem 30 when  $a = 0.2$ ,  $b = 2.43$ ,  $c = 1.61$  and  $d = 0.018455$ . The case when  $E_{SW}$  is a repeller,  $\{P, T(P)\}$  is the period two-solution which is non-hyperbolic ( $\mu_1 = 1$  and  $\mu_2 = 0.567822$ ) and  $\{P_2, T(P_2)\}$  is the period two-solution which is a saddle point.

$\mathcal{C}(P), \mathcal{C}(T(P))$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. The first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of four disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_2, T(P_2)\}) \cup \mathcal{B}(\{P, T(P)\}) \cup \mathcal{B}(E_{NE}),$$

where

$$\mathcal{B}(\{P_2, T(P_2)\}) = \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2)),$$

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\},$$

$$\mathcal{B}(E_{NE}) = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))\},$$

$$\mathcal{B}(\{P, T(P)\}) = \mathcal{C}(P) \cup \mathcal{C}(T(P)) \cup \{(x, y) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x, y) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}(P) \cup \mathcal{C}(T(P))$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\}.$$

Thus, I have

$$\mathcal{W}^s(P_2) = \mathcal{C}_1^+, \mathcal{W}^s(T(P_2)) = \mathcal{C}_1^-, \mathcal{C}(P) = \mathcal{C}_2^+, \text{ and } \mathcal{C}(T(P)) = \mathcal{C}_2^-.$$

**The case  $\Delta \geq 0$**

**Theorem 31** Assume that  $\Delta > 0$ . Then  $E_0(0, 0)$  is globally asymptotically stable and  $B(E_0) = [0, \infty)^2$ .

**Proof.** By Lemma 20 I have  $T([0, \infty)^2) \subseteq [0, U_1] \times [0, U_2]$ , so it is sufficient to prove that  $T^n(x_0, y_0) \rightarrow E_0$  for  $(x_0, y_0) \in [0, U_1] \times [0, U_2]$ . Take  $(x_0, y_0) \in [0, U_1] \times [0, U_2]$ . From  $T((U_1, U_2)) \preceq_{ne} (U_1, U_2)$ , since  $T$  is a cooperative map, I obtain  $T^{n+1}(U_1, U_2) \preceq_{ne} T^n(U_1, U_2)$ . By Lemma 17,  $E_0$  is the only equilibrium point so I have that  $T^n(U_1, U_2) \rightarrow E_0$  as  $n \rightarrow \infty$ . Further, from  $(x_0, y_0) \preceq_{ne} (U_1, U_2)$  I have  $E_0 \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(U_1, U_2)$ . This implies that  $T^n(x_0, y_0) \rightarrow E_0$  as  $n \rightarrow \infty$ . The same proof can be accomplished by using Theorem 40.

See Figure 7 b) for the visual illustration. □

**Theorem 32** If  $\Delta = 0$  and the minimal period-two solution does not exist, then System (55) has two equilibrium points  $E_0 \ll E$ , where  $E_0$  is locally asymptotically

stable and  $E$  is non-hyperbolic and there exist a continuous curve  $\mathcal{C}^s(E)$  passing through the point  $E$ , such that  $\mathcal{C}^s(E)$  is a graph of decreasing function. The first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of two disjoint basins of attraction, namely  $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$ , where

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \prec_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}^s(E)\},$$

$$\mathcal{B}(E) = \{(x, y) | (x, y) \preceq_{ne} (x_E, y_E) \text{ for some } (x_E, y_E) \in \mathcal{C}^s(E)\}.$$

**Proof.** Lemma 17 implies that there exist two equilibrium solutions namely  $E_0$ ,  $E$  such that  $E_0 \ll_{ne} E$ . By Theorem 21 the equilibrium solution  $E_0$  is locally asymptotically stable. By Theorem 22 the equilibrium solution  $E$  is non-hyperbolic of stable type, i.e.  $\lambda = 1$  and  $-1 < \mu < 1$ . In view of (70) eigenvector associated with  $\mu$  is not a coordinate axis. In view of (66) the map  $T$  is strongly cooperative on  $[0, \infty)^2$ . Hence, all conditions of Theorems 1 and 4 [66] for cooperative map  $T$  are satisfied, which yields the existence of the invariant curve  $\mathcal{C}_E$  which is passing through the point  $E$ , and it is graph of decreasing function. Let

$$\mathcal{W}^- = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0}) \text{ for some } (x_{E_0}, y_{E_0}) \in \mathcal{C}_E\},$$

$$\mathcal{W}^+ = \{(x, y) | (x, y) \preceq_{ne} (x_{E_{NE}}, y_{E_{NE}}) \text{ for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{C}_E\}.$$

Take  $(x_0, y_0) \in \mathcal{W}^- \cap [0, \infty)^2$  and  $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{W}^+ \cap [0, \infty)^2$ . By Theorem 4 [66] I have that there exists  $n_0 > 0$  such that,  $T^n(x_0, y_0) \in \text{int}(Q_3(E))$  and  $T^n(\tilde{x}_0, \tilde{y}_0) \in \text{int}(Q_1(E))$  for  $n > n_0$ . The rest of the proof follows from Lemma 20. See Figure 7 a) for its visual illustration.  $\square$

**Remark 5** Theorems 26-32 can be generalized to a general cooperative system

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots, \quad (74)$$

where  $f, g$  are increasing functions in their variables. Indeed the proofs of Theorems 26-32 are based on general results for cooperative systems and did not use the

specific structure of System (55). In fact any cooperative system with the same configuration and local character of its equilibrium and period-two solution would have the same dynamics. For instance Theorem 28 has the following immediate generalization.

**Theorem 33** *Assume that a cooperative system (74) has three equilibrium solutions  $E_0 \ll E_{SW} \ll E_{NE}$ , where  $E_0$  and  $E_{NE}$  are locally asymptotically stable and  $E_{SW}$  is repeller or non-hyperbolic equilibrium point and that every solution of (74) is bounded. Further assume that (74) has three minimal period-two solutions  $\{P_i, T(P_i)\}$   $i = 1, 2, 3$ , where  $\{P_1, T(P_1)\}$  and  $\{P_2, T(P_2)\}$  are the saddle points and  $\{P_3, T(P_3)\}$  is locally asymptotically stable. In this case there exist four continuous curves  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2))$  where  $\mathcal{W}^s(P_1), \mathcal{W}^s(T(P_1)), \mathcal{W}^s(P_2), \mathcal{W}^s(T(P_2))$  are passing through the point  $E_{SW}$ , and are graphs of decreasing functions. Every solution which starts below  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  in the North-east ordering converges to  $E_0$  and every solution which starts above  $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$  in the North-east ordering converges to  $E_{NE}$ . Every solution which starts above  $\mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))$  and below  $\mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$  in the North-east ordering converges to  $\{P_3, T(P_3)\}$ . In other words, the first quadrant of initial condition  $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$  is the union of five disjoint basins of attraction, namely*

$$Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1, T(P_1)\}) \cup \mathcal{B}(\{P_2, T(P_2)\}) \cup \mathcal{B}(\{P_3, T(P_3)\}) \cup \mathcal{B}(E_{NE}),$$



where

$$\mathcal{B}(\{P_1, T(P_1)\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1)),$$

$$\mathcal{B}(\{P_2, T(P_2)\}) = \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2)),$$

$$\mathcal{B}(E_0) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_0}, y_{E_0})$$

$$\text{for some } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\},$$

$$\mathcal{B}(E_{NE}) = \{(x, y) | (x_{E_{NE}}, y_{E_{NE}}) \preceq_{ne} (x, y)$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))\},$$

$$\mathcal{B}(\{P_3, T(P_3)\}) = \{(x, y) | (x_{E_0}, y_{E_0}) \preceq_{ne} (x, y) \preceq (x_{E_{NE}}, y_{E_{NE}}),$$

$$\text{for some } (x_{E_{NE}}, y_{E_{NE}}) \in \mathcal{W}^s(P_2) \cup \mathcal{W}^s(T(P_2))$$

$$\text{and } (x_{E_0}, y_{E_0}) \in \mathcal{W}^s(P_1) \cup \mathcal{W}^s(T(P_1))\}.$$

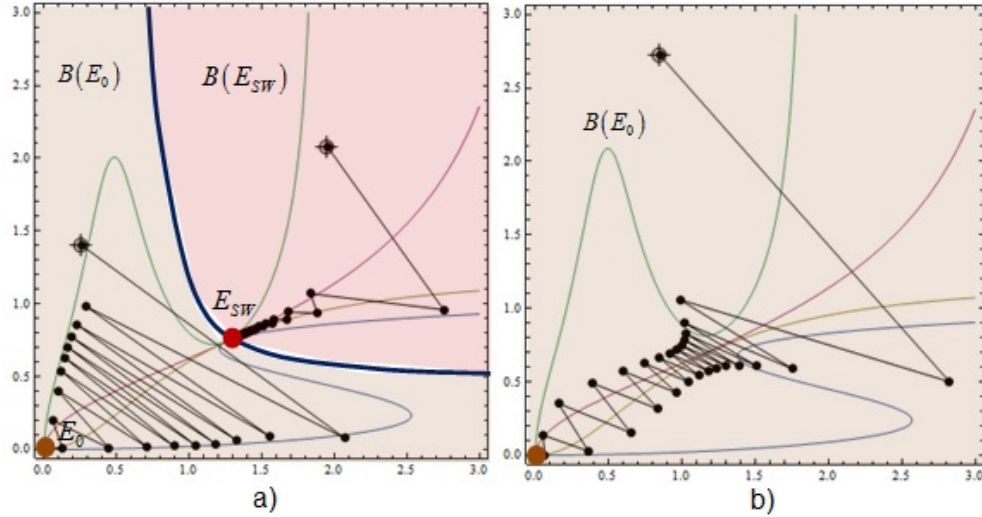


Figure 9. a) The case when  $E_{SW}$  is non-hyperbolic. b) The case when there exists only one equilibrium point  $E_0$ .

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MANUSCRIPT 4

**Global Asymptotic Stability for Discrete Single Species Biological  
Models**

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## 4.1 Introduction

The following difference equation is known as Beverton-Holt model

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, \dots \quad (75)$$

where  $a > 0$  is a rate of change (growth or decay) and  $x_n$  is the size of population at  $n$ -th generation. It was introduced by Beverton and Holt in 1957 and depicts density dependent recruitment of a population with limited resources in which resources are not shared equally. It assumes that the per capita number of offspring is inversely proportional to a linearly increasing function of the number of adults.

Let  $P(n)$  be the size of a population in generation  $n$ .

Suppose that  $\mu$  is the net reproductive rate, that is the number of offspring that each individual leaves before dying if there are no limitation in resources. Then the Beverton-Holt model is given by

$$\frac{P(n+1)}{P(n)} = \frac{\mu}{1 + \frac{\mu-1}{K}P(n)}$$

where  $K > 0$  is carrying capacity. This leads to the equation

$$P(n+1) = \frac{\mu P(n)}{1 + \frac{\mu-1}{K}P(n)}, \quad n = 0, 1, \dots \quad (76)$$

which by the substitution  $x_n = \frac{\mu-1}{K}P(n)$  reduces to Equation (75) where  $a = \mu$ .

The model is well studied and understood and exhibits the following properties:

**Theorem 34** *Equation (75) has the following properties:*

1. *Equation (75) has two equilibrium points 0 and  $a - 1$  when  $a > 1$ .*
2. *All solutions of Equation (75) are monotonic (increasing or decreasing sequences).*

3. If  $a \leq 1$ , then the equilibrium point 0 is the **global attractor** (that is  $\lim_{n \rightarrow \infty} x_n = 0$ ).
4. If  $a > 1$ , then the equilibrium point  $a - 1$  is the global attractor (that is  $\lim_{n \rightarrow \infty} x_n = a - 1$ , when  $x_0 > 0$ ).
5. Both equilibrium points are **globally asymptotically stable** that is, they are global attractors with the property that small changes of initial condition  $x_0$  result in small changes of the corresponding solution  $\{x_n\}$ .
6. Equation (75) can be solved explicitly and has the following solution:

$$\begin{aligned} x_n &= \frac{1}{1/(a-1) + (1/x_0 - 1/(a-1))1/a^n} \quad \text{if } a \neq 1 \\ x_n &= \frac{1}{n+1/x_0}, \quad \text{if } a = 1, \end{aligned} \tag{77}$$

which implies all the above-mentioned properties of Equation (75).

See for example [5, 14, 24, 84, 87].

The following difference equation is known as **Beverton-Holt model with constant immigration**

$$x_{n+1} = \frac{ax_n}{1 + x_n} + h, \quad n = 0, 1, \dots \tag{78}$$

where  $a > 0$  is a rate of change (growth or decay),  $h > 0$  is a constant immigration and  $x_n$  is the size of population at  $n$ -th generation. The simple substitution  $y_n = x_n - h$  reduces Equation (78) to the so-called **Riccati's equation**

$$y_{n+1} = \frac{ay_n + ah}{y_n + 1 + h}, \quad n = 0, 1, \dots \tag{79}$$

which is well studied and understood and exhibits the following properties:

**Theorem 35**    1. Equation (78) has one positive equilibrium point  $\bar{y}$ .

2. All solutions of Equation (78) are monotonic (increasing or decreasing sequences) and bounded.
3. The equilibrium point  $\bar{y}$  is a global attractor and is globally asymptotically stable.
4. Furthermore, Equation (78) can be solved explicitly and has the following solution:

$$y_n = (a + 1 + h) \frac{(w_0 - w_-)w_+^{n+1} - (w_+ - w_0)w_-^{n+1}}{(w_0 - w_-)w_+^n - (w_+ - w_0)w_-^n} - 1 - h, \quad (80)$$

where  $w_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 - 4\mathcal{R}})$  and  $\mathcal{R} = \frac{a}{(a+1+h)^2}$  and

$$w_0 = \frac{y_0 + 1 + h}{a + 1 + h}.$$

The biological implications of this model are that the constant immigration eliminates the possibility of zero equilibrium and so all solutions get attracted to the unique positive equilibrium solution.

The Beverton-Holt model with emigration leads to the equation

$$x_{n+1} = \frac{ax_n}{1 + x_n} - h, \quad n = 0, 1, \dots \quad (81)$$

where  $a > 0$  is a rate of change (growth or decay),  $h > 0$  is a constant emigration. The solution of Equation (81) is given by (80) where  $h$  should be replaced by  $-h$ .

Equation (81) has quite different dynamics than Equation (78), since it can have 0, 1 or 2 equilibrium solutions, depending on the values of the expression

$$D = (1 + h - a)^2 - 4h. \quad (82)$$

The following results holds for Equation (81).

**Theorem 36** *Equation (81) exhibits one of the following three scenarios:*

1. If  $D < 0$  then every solution of Equation (81) satisfies  $\lim_{n \rightarrow \infty} x_n = \infty$ . In addition, every solution is increasing.;
2. If  $D = 0$  then Equation (81) has a single equilibrium  $\bar{x}$  which is nonhyperbolic and every solution of Equation (81) satisfies  $\lim_{n \rightarrow \infty} x_n = -\infty$  if  $x_0 < \bar{x}$  and  $\lim_{n \rightarrow \infty} x_n = \bar{x}$  if  $x_0 > \bar{x}$ . In addition, every solution is decreasing.
3. If  $D > 0$  then Equation (81) has two positive equilibrium solutions  $\bar{x}_- < \bar{x}_+$  where  $\bar{x}_-$  is a repeller and  $\bar{x}_+$  is locally asymptotically stable. Every solution of Equation (81) satisfies  $\lim_{n \rightarrow \infty} x_n = -\infty$  if  $x_0 < \bar{x}_-$  and  $\lim_{n \rightarrow \infty} x_n = \bar{x}_+$  if  $x_0 > \bar{x}_-$ . In addition, every solution which starts in  $(\bar{x}_-, \bar{x}_+)$  is increasing, while the other non-constant solutions are decreasing.

The biological implications of this model are that the constant emigration introduces the possibility of the threshold such that if the initial population is below that threshold the population goes to extinction.

See Kulenović, Ladas [54] and Kulenović, Merino [60].

The following difference equation is known as Beverton-Holt model with periodic immigration

$$x_{n+1} = \frac{ax_n}{1 + x_n} + h_n, \quad n = 0, 1, \dots \quad (83)$$

where  $a > 0$  is a rate of change (growth or decay),  $h_n > 0$  is a periodic immigration (sequence) and  $x_n$  is the size of population at  $n$ -th generation. The substitution  $y_n = x_n - h_n$  reduces Equation (83) to the so-called **Riccati's equation** with periodic coefficients

$$y_{n+1} = \frac{ay_n + ah_n}{y_n + 1 + h_n}, \quad n = 0, 1, \dots \quad (84)$$

which is, very recently, studied and understood and exhibits the following properties, see [35]:



**Theorem 37** *Equation (83) has the following properties:*

1. *Equation (83) has the unique non-negative periodic solution  $\bar{x}_n$ , which period equals this of  $h_n > 0$ .*
2. *The periodic solution  $\{\bar{x}_n\}$  is the global attractor and all solutions of Equation (83) are attracted to that periodic solution.*
3. *There is a procedure for finding the explicit solution of Equation (83). In particular, there are explicit formulas for the cases when  $h_n$  is periodic sequence with periods 2, 3, 4.*

The biological implications of model (83) are that the periodic immigration imposes its periodicity on the solutions of the model and so all solutions get attracted to the unique periodic solution.

Case of periodic emigration is quite different as this emigration may introduce the periodic threshold which would imply the extinction scenario if the initial population is below that threshold.

See Grove, Kostrov, Ladas, Schultz [35].

The following difference equation is known as the Beverton-Holt model with periodic environment

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, \quad n = 0, 1, \dots \quad (85)$$

where  $\mu > 1$  is a rate of change (growth or decay),  $K_n > 0$  is a periodic sequence of period  $p$  modeling periodicity of environment (periodic supply of food, energy etc.) and  $x_n$  is the size of population at  $n$ -th generation.

Rewriting Equation (85) as

$$\frac{1}{x_{n+1}} = \frac{K_n + (\lambda - 1)x_n}{\lambda K_n x_n},$$

the substitution  $y_n = 1/x_n$  reduces Equation (85) to the linear non-autonomous equation

$$y_{n+1} = \frac{1}{\mu}y_n + p_n, \quad n = 0, 1, \dots \quad (86)$$

where  $p_n = \frac{\mu-1}{\mu K_n}$ . The solution of Equation (86) is given as

$$y_n = \frac{1}{\mu^n}y_0 + \sum_{k=0}^{n-1} \frac{1}{\mu^{n-k-1}}p_k$$

and it is well studied and understood and exhibits the following properties:

**Theorem 38** *Equation (86) has the following properties:*

1. *Equation (86) has the unique non-negative periodic solution  $\bar{y}_n$ , with period equals to  $p$ .*
2. *The periodic solution  $\{\bar{y}_n\}$  is the global attractor and all solutions of Equation (86) are attracted to that periodic solution.*
3. *The **periodic environment is deleterious** in the sense that the size of population in periodic environment is smaller than the average of sizes in  $p$  constant environments. I say that in this case the periodic solution is an **attenuant cycle**. Mathematically, this means that*

$$\frac{1}{p}(\bar{y}_1 + \dots + \bar{y}_p) < \frac{1}{p}((K_1 - 1) + \dots + (K_p - 1))$$

*where  $K_i - 1$  is the equilibrium of Equation (75) when  $a = K_i$ .*

This is an example of so-called **Jillson's effect** that refers to any change in global behavior caused by a periodic fluctuation of the environment.

See [28, 29].

The following difference equation, known as sigmoid Beverton-Holt model, is mathematically the simplest Beverton-Holt type model that exhibits the Allee's

effect

$$x_{n+1} = \frac{ax_n^2}{1 + x_n^2}, \quad n = 0, 1, \dots \quad (87)$$

where  $a > 0$  is a rate of change (growth or decay) and  $x_n$  is the size of population at  $n$ -th generation. The model is well studied and understood and exhibits the following properties:

- Theorem 39**    1. Equation (75) has either one equilibrium point 0 when  $a < 2$ , or two equilibrium points 0 and 1, when  $a = 2$ , or three equilibrium points 0 and  $\bar{x}_- = \frac{a-\sqrt{a^2-4}}{2} < \bar{x}_+ = \frac{a+\sqrt{a^2-4}}{2}$  when  $a > 2$ .
2. All solutions of Equation (75) are monotonic (increasing or decreasing sequences) and bounded.
3. If  $a < 2$ , then the equilibrium point 0 is the global attractor.
4. If  $a = 2$ , then the equilibrium point 0 is the attractor with **the basin of attraction**  $\mathcal{B}(0) = [0, 1)$  and 1 is nonhyperbolic with the basin of attraction  $\mathcal{B}(1) = (1, \infty)$ .
5. If  $a > 2$ , then there are two attractors: 0 with the basin of attraction  $\mathcal{B}(0) = [0, \bar{x}_-)$  and  $\bar{x}_+$  with the basin of attraction  $\mathcal{B}(\bar{x}_+) = (\bar{x}_-, \infty)$ .

The biological implications of this model is that exhibits so-called **Allee's effect** ( the social dysfunction and failure to mate successfully when population density falls below a certain threshold) in the sense that if the initial size  $x_0$  is smaller than  $\bar{x}_-$  the population goes to extinction.

## 4.2 Preliminaries

In this part I present basic tools which I use to extend the results in Section 4.1 to more general models that includes several age groups or several species.

#### 4.2.1 Global attractivity results for monotone systems

Let  $\preceq$  be a partial order on  $\mathbb{R}^n$  with nonnegative cone  $P$ . For  $x, y \in \mathbb{R}^n$  the *order interval*  $\llbracket x, y \rrbracket$  is the set of all  $z$  such that  $x \preceq z \preceq y$ . I say  $x \prec y$  if  $x \preceq y$  and  $x \neq y$ , and  $x \ll y$  if  $y - x \in \text{int}(P)$ . A map  $T$  on a subset of  $\mathbb{R}^n$  is *order preserving* if  $T(x) \preceq T(y)$  whenever  $x \prec y$ , *strictly order preserving* if  $T(x) \prec T(y)$  whenever  $x \prec y$ , and *strongly order preserving* if  $T(x) \ll T(y)$  whenever  $x \prec y$ .

Let  $T : R \rightarrow R$  be a map with a fixed point  $\bar{x}$  and let  $R'$  be an invariant subset of  $R$  that contains  $\bar{x}$ . I say that  $\bar{x}$  is *stable* (asymptotically stable) relative to  $R'$  if  $\bar{x}$  is a stable (asymptotically stable) fixed point of the restriction of  $T$  to  $R'$ .

The next result in [67] is stated for order-preserving maps on  $\mathbb{R}^n$ . See [21] for a more general version valid in ordered Banach spaces.

**Theorem 40** *For a nonempty set  $R \subset \mathbb{R}^n$  and  $\preceq$  a partial order on  $\mathbb{R}^n$ , let  $T : R \rightarrow R$  be an order preserving map, and let  $a, b \in R$  be such that  $a \prec b$  and  $\llbracket a, b \rrbracket \subset R$ . If  $a \preceq T(a)$  and  $T(b) \preceq b$ , then  $\llbracket a, b \rrbracket$  is an invariant set and*

- i. *There exists a fixed point of  $T$  in  $\llbracket a, b \rrbracket$ .*
- ii. *If  $T$  is strongly order preserving, then there exists a fixed point in  $\llbracket a, b \rrbracket$  which is stable relative to  $\llbracket a, b \rrbracket$ .*
- iii. *If there is only one fixed point in  $\llbracket a, b \rrbracket$ , then it is a global attractor in  $\llbracket a, b \rrbracket$  and therefore asymptotically stable relative to  $\llbracket a, b \rrbracket$ .*

The following result in [67] is a direct consequence of the trichotomy result of Dancer and Hess in [21].

**Corollary 5** *If the nonnegative cone of  $\preceq$  is a generalized quadrant in  $\mathbb{R}^n$ , and if  $T$  has no fixed points in  $\llbracket u_1, u_2 \rrbracket$  other than  $u_1$  and  $u_2$ , then the interior of  $\llbracket u_1, u_2 \rrbracket$  is either a subset of the basin of attraction of  $u_1$  or a subset of the basin of attraction of  $u_2$ .*

Consider the general  $k$ -th order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k+1}), \quad n = 0, 1, \dots, \quad (88)$$

where  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is continuous function. Introducing new variables

$$u_n^1 = x_{n-k+1}, u_n^2 = x_{n-k+2}, \dots, u_n^k = x_n$$

I rewrite Equation (88) as the system

$$u_{n+1}^1 = u_n^2, u_{n+1}^2 = u_n^3, \dots, u_{n+1}^{k-1} = u_n^k, u_{n+1}^k = f(u_n^k, u_n^{k-1}, \dots, u_n^1) \quad (89)$$

which corresponding map has the form

$$T\left(\begin{pmatrix} u^1 \\ u^2 \\ \dots \\ u^{k-1} \\ u^k \end{pmatrix}\right) = \begin{pmatrix} u^2 \\ u^3 \\ \dots \\ u^k \\ f(u^k, \dots, u^1) \end{pmatrix}. \quad (90)$$

The map  $T$  is non-decreasing map with respect to the ordering  $\preceq$  in  $\mathbb{R}^k$  defined as:

$$\mathbf{u} \preceq \mathbf{v} \Leftrightarrow u_i \leq v_i, i = 1, \dots, k,$$

where  $\mathbf{u} = (u_1, \dots, u_k), \mathbf{v} = (v_1, \dots, v_k)$ , in the sense that

$$\mathbf{u} \preceq \mathbf{v} \Rightarrow T(\mathbf{u}) \preceq T(\mathbf{v}),$$

for every  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$ . Set  $\mathbf{0} = (0, 0, \dots, 0), \mathbf{x}_- = (x_-, x_-, \dots, x_-)$ . The interval  $[\mathbf{0}, \mathbf{x}_-] = \{\mathbf{x} : \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{x}_-\}$  is invariant set for the map  $T$ , that is  $T([\mathbf{0}, \mathbf{x}_-]) \subset [\mathbf{0}, \mathbf{x}_-]$ . Consequently, by Corollary 5, the interior of the interval  $[\mathbf{0}, \mathbf{x}_-]$  is a part of the basin of attraction of one of two fixed points  $\mathbf{0}, \mathbf{x}_-$ . Since  $\mathbf{0}$  is locally asymptotically stable the interior of the interval  $[\mathbf{0}, \mathbf{x}_-]$  is a part of the basin of attraction of  $\mathbf{0}$ .

The reasoning given in the above discussion can be extended to the case of general difference equation (88) to give the following result.

**Theorem 41** *Consider Equation (88) where  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  is a continuous, non-decreasing in all variables and bounded function with the lower bound  $L$  and upper bound  $U$  respectively. If Equation (88) has two equilibrium points  $x_L < x_U$ , such that  $x_L$  is unstable and  $x_U$  is asymptotically stable, then the equilibrium  $x_L$  is globally asymptotically stable within its basin of attraction which contains  $[L, x_L]^{k+1}$  and the equilibrium  $x_U$  is globally asymptotically stable within its basin of attraction which contains  $(x_L, x_U]^{k+1} \cup [x_U, U]^{k+1}$ .*

#### 4.2.2 Difference inequalities

In this section I give some basic results on difference inequalities which I will use later to extend some of our results for autonomous equation to the case of asymptotically autonomous difference equations. The results extend some of the results in [49, 75].

**Theorem 42** *Let  $n \in N_{n_0}^+$  and  $g(n, u, v)$  be a nondecreasing function in  $u$  and  $v$  for any fixed  $n$ . Suppose that for  $n \geq n_0$ , the inequalities*

$$y_{n+1} \leq g(n, y_n, y_{n-1}) \quad (91)$$

$$u_{n+1} \geq g(n, u_n, u_{n-1}) \quad (92)$$

*hold. Then*

$$y_{n_0-1} \leq u_{n_0-1}, \quad y_{n_0} \leq u_{n_0} \quad (93)$$

*implies that*

$$y_n \leq u_n, \quad n \geq n_0. \quad (94)$$

**Proof.** Suppose that (94) is not true. Then, there exists a smallest  $k \in N_{n_0}^+$  such that

$$y_{k+1} > u_{k+1}.$$

By using Equation (91), (92) and (93) and the monotone character of  $g$ , it follows from

$$y_k \leq u_k, \quad y_{k-1} \leq u_{k-1}$$

that

$$y_{k+1} \leq g(k, y_k, y_{k-1}) \leq g(k, u_k, u_{k-1}) \leq u_{k+1},$$

which is a contradiction.  $\square$

Applying Theorem 42 twice, I obtain the following result.

**Corollary 6** *Suppose that  $g_1(n, u, v)$  and  $g_2(n, u, v)$  are two functions defined on  $N_{n_0}^+ \times \mathbb{R}^2$  and nondecreasing with respect to  $u$  and  $v$ . Let*

$$g_2(n, u_n, u_{n-1}) \leq u_{n+1} \leq g_1(n, u_n, u_{n-1}).$$

*Then*

$$L_n \leq u_n \leq U_n$$

*where  $L_n$  and  $U_n$  are the solutions of the difference equations*

$$U_{n+1} = g_1(n, U_n, U_{n-1}), \quad U_{n_0} \geq u_{n_0}, \quad U_{n_0-1} \geq u_{n_0-1}$$

$$L_{n+1} = g_2(n, L_n, L_{n-1}), \quad L_{n_0} \leq u_{n_0}, \quad L_{n_0-1} \leq u_{n_0-1}$$

An immediate extension of Theorem 42 is the following result:

**Theorem 43** *Let  $y_i$   $n=0,1,2,\dots$ , be sequences satisfying*

$$y_{n+1} \leq g_1(n, y_n, \dots, y_{n-p})$$

$$u_{n+1} \geq g_2(n, u_n, \dots, u_{n-p}),$$

*where  $g_i$  is nondecreasing with respect to its argument. Then,*

$$y_{n_0-p} \leq u_{n_0-p}, \dots, \quad y_{n_0-1} \leq u_{n_0-1}, \quad y_{n_0} \leq u_{n_0}$$

*implies*

$$y_n \leq u_n, \quad n \geq n_0.$$

**Theorem 44** *Consider the difference equation*

$$x_{n+1} = a_n f(x_n), \quad n = 0, 1, \dots, \quad (95)$$

*where  $f$  is nondecreasing function and*

$$\lim_{n \rightarrow \infty} a_n = a, \quad (96)$$

*and the limiting difference equation*

$$y_{n+1} = a f(y_n), \quad n = 0, 1, \dots \quad (97)$$

*Assume that there exists  $\epsilon_0 > 0$  such that every solution of difference equation*

$$y_{n+1} = A f(y_n), \quad n = 0, 1, \dots \quad (98)$$

*converges to a constant solution  $\bar{y}_A$  for every  $A \in (a - \epsilon_0, a + \epsilon_0)$ . If*

$$\lim_{A \rightarrow a} \bar{y}_A = \bar{y}, \quad (99)$$

*then every solution of the difference equation (95) satisfies*

$$\lim_{n \rightarrow \infty} x_n = \bar{y}. \quad (100)$$

**Proof.** In view of (97) for every  $\epsilon > 0$  there exists  $N = N(\epsilon) > 0$  such that

$$n \geq N(\epsilon) \implies |a_n - a| < \epsilon \iff (a - \epsilon) < a_n < (a + \epsilon),$$

which implies

$$n \geq N(\epsilon) \implies (a - \epsilon) f(x_n) \leq x_{n+1} = a_n f(x_n) < (a + \epsilon).$$

Now, assume that  $\epsilon \leq \epsilon_0$  and consider two equations of the form (98), where  $A = a - \epsilon$  and  $A = a + \epsilon$ . By the Corollary 2 from [75] I have that

$$\ell_n \leq x_n \leq u_n, \quad n \geq N(\epsilon), \quad (101)$$



where  $\{\ell_n\}$  satisfies

$$\ell_{n+1} = (a - \epsilon)f(\ell_n) \quad n \geq N(\epsilon)$$

and  $\{u_n\}$  satisfies

$$u_{n+1} = (a + \epsilon)f(u_n) \quad n \geq N(\epsilon).$$

In view of the assumptions

$$\lim_{n \rightarrow \infty} \ell_n = \lim_{n \rightarrow \infty} u_n = \bar{y},$$

which completes the proof.  $\square$

**Example 8** The difference equation

$$y_{n+1} = (1 + a_n)y_n, \quad n = 0, 1, \dots \quad (102)$$

where  $a_n \geq 0, n = 0, 1, \dots$  and  $y_0 \geq 0$  has a solution

$$y_n = \prod_{k=0}^{n-1} (1 + a_k)x_0,$$

that is convergent if and only if  $\sum_{k=0}^{\infty} a_k$  is convergent. In this case  $\lim_{k \rightarrow \infty} a_k = 0$  and the limiting equation is

$$x_{n+1} = x_n, \quad n = 0, 1, \dots \quad (103)$$

Similarly,  $\sum_{k=0}^{\infty} a_k$  can be divergent and yet  $\lim_{k \rightarrow \infty} a_k = 0$  with the limiting equation (103). This shows that if the limiting equation is non-hyperbolic the dynamics of original equation can be very diverse.

**Example 9** The difference equation

$$y_{n+1} = b_n \frac{y_n}{1 + y_n}, \quad n = 0, 1, \dots \quad (104)$$

where  $b_n > 0, n = 0, 1, \dots, \lim_{n \rightarrow \infty} b_n = 1$  and  $y_0 > 0$  can be transformed into equation

$$u_{n+1} = B_n(1 + u_n), \quad n = 0, 1, \dots, \quad (105)$$

where  $B_n = 1/b_n$ ,  $u_n = 1/y_n$ . Solving equation (105) and going back to  $y_n$  I obtain

$$y_n = \left( \prod_{k=0}^{n-1} (1 + a_k) \frac{1}{y_0} + \sum_{k=0}^{\infty} \prod_{i=k}^{n-1} (1 + a_i) \right)^{-1}$$

that is convergent if and only if  $\sum_{k=0}^{\infty} a_k$  and  $\sum_{k=0}^{\infty} \prod_{i=k}^{n-1} (1 + a_i)$  are convergent, where  $\frac{1}{b_i} = 1 + a_i$ ,  $i = 1, 2, \dots$

In this case  $\lim_{k \rightarrow \infty} a_k = 0$  and the limiting equation is

$$x_{n+1} = \frac{x_n}{1 + x_n}, \quad n = 0, 1, \dots \quad (106)$$

This shows that if the limiting equation is non-hyperbolic the dynamics of original equation can be very diverse.

**Example 10** The difference equation (104) where  $b_n > 0$ ,  $n = 0, 1, \dots$ ,  $\lim_{n \rightarrow \infty} b_n = a$  and  $y_0 > 0$ , has simple behavior in the hyperbolic case, that is when  $a \neq 1$ . Indeed Theorem 44 implies that in this case the global dynamics of Equation (104) is the same as the global dynamics of the limiting equation (75), described in Theorem 34. Thus the following result holds

$$\lim_{n \rightarrow \infty} b_n = a \begin{cases} < 1 \\ > 1 \end{cases} \implies \lim_{n \rightarrow \infty} x_n = \begin{cases} 0, & \text{if } x_0 \geq 0 \\ a - 1, & \text{if } x_0 > 0. \end{cases}$$

Similarly an application of Theorem 44 gives the following result:

**Example 11** Consider the difference equation

$$x_{n+1} = a_n \frac{x_n^2}{1 + x_n^2}, x_0 \in \mathbb{R} \quad a_n \geq 0, \quad n = 0, 1, \dots \quad (107)$$

where (96) holds. Then the following result holds

$$\lim_{n \rightarrow \infty} a_n = a \begin{cases} < 2 \\ > 2 \end{cases} \implies \lim_{n \rightarrow \infty} x_n = \begin{cases} 0, & \text{if } 0 \leq x_0 < \bar{x}_- \\ \bar{x}_+, & \text{if } x_0 > \bar{x}_- \end{cases}$$

where  $\bar{x}_- < \bar{x}_+$  are the positive equilibrium solutions of the corresponding limiting equation

$$x_{n+1} = a \frac{x_n^2}{1 + x_n^2}, \quad n = 0, 1, \dots \quad (108)$$

In this case difference equation exhibits the Allee's Effect.

**Theorem 45** *Consider the difference equation*

$$x_{n+1} = a_n f_1(x_n) + b_n f_2(x_{n-1}), \quad n = 0, 1, \dots \quad (109)$$

where  $f_i, \quad n = 0, 1, \dots$  are nondecreasing functions and

$$\lim_{n \rightarrow \infty} a_n = a, \quad \lim_{n \rightarrow \infty} b_n = b, \quad (110)$$

and the limiting difference equation

$$y_{n+1} = a f_1(y_n) + b f_2(y_{n-1}), \quad n = 0, 1, \dots \quad (111)$$

Assume that there exists  $\epsilon_0 > 0$  such that every solution of difference equation

$$y_{n+1} = A f_1(y_n) + B f_2(y_{n-1}), \quad n = 0, 1, \dots$$

converges to a constant solution  $\bar{y}_{A,B}$  for every  $A \in (a - \epsilon_0, a + \epsilon_0)$  and  $B \in (b - \epsilon_0, b + \epsilon_0)$ . If

$$\lim_{A \rightarrow a, B \rightarrow b} \bar{y}_{A,B} = \bar{y}, \quad (112)$$

where  $\bar{y}$  is an equilibrium solution of the limiting difference equation (111), then every solution of the difference equation (109) satisfies

$$\lim_{n \rightarrow \infty} x_n = \bar{y}. \quad (113)$$

**Proof.** In view of (110) for all  $\epsilon > 0$  there exists  $N_1(\epsilon) > 0$  and  $N_2(\epsilon) > 0$  such that

$$n \geq N_1(\epsilon) \implies |a_n - a| < \epsilon \iff (a - \epsilon) < a_n < (a + \epsilon),$$

$$n \geq N_2(\epsilon) \implies |b_n - b| < \epsilon \iff (b - \epsilon) < b_n < (b + \epsilon).$$

Let  $N = \max(N_1, N_2)$ . Then  $n \geq N(\epsilon)$  implies

$$(a - \epsilon)f_1(x_n) + (b - \epsilon)f_2(x_{n-1}) \leq x_{n+1} = a_nf_1(x_n) + b_nf_2(x_{n-1}) < (a + \epsilon)f_1(x_n) + (b + \epsilon)f_2(x_{n-1}).$$

Now, assume that  $\epsilon \leq \epsilon_0$  and consider two equations of the form (111). In view of Corollary 6 I have that

$$\ell_n \leq x_n \leq u_n, \quad n \geq N(\epsilon), \quad (114)$$

where  $\{\ell_n\}, \{u_n\}$  satisfies

$$\begin{cases} \ell_{n+1} = (a - \epsilon)f_1(\ell_n) + (b - \epsilon)f_2(\ell_{n-1}), & n \geq N(\epsilon) \\ u_{n+1} = (a + \epsilon)f_1(u_n) + (b + \epsilon)f_2(u_{n-1}), & n \geq N(\epsilon). \end{cases} \quad (115)$$

In view of the assumption (112) I have that

$$\lim_{n \rightarrow \infty} \ell_n = \lim_{n \rightarrow \infty} u_n = \bar{y},$$

which by (114) implies (113).  $\square$

Theorem 45 has an immediate extension to the  $k$ -th order difference equation of the form

$$x_{n+1} = \sum_{i=0}^{k-1} a_i(n)f_i(x_{n-i}), \quad n = 0, 1, \dots \quad (116)$$

**Theorem 46** *Consider the difference equation (116) where  $f_i, \quad i = 0, 1, \dots, k-1$  are nondecreasing functions and*

$$\lim_{n \rightarrow \infty} a_i(n) = \alpha_i, \quad i = 0, \dots, k-1 \quad (117)$$

*and the limiting difference equation*

$$y_{n+1} = \sum_{i=0}^{k-1} \alpha_i f_i(x_{n-i}), \quad n = 0, 1, \dots \quad (118)$$

*Assume that there exists  $\epsilon_0 > 0$  such that every solution of difference equation*

$$y_{n+1} = \sum_{i=0}^{k-1} A_i f_i(x_{n-i}), \quad n = 0, 1, \dots$$

converges to a constant solution  $\bar{y}_{\alpha_i}$  for every  $A_i \in (\alpha_i - \epsilon_0, \alpha_i + \epsilon_0)$ . If

$$\lim_{A_i \rightarrow \alpha_i, i=0, \dots, k} \bar{y}_{A_i} = \bar{y}, \quad (119)$$

where  $\bar{y}$  is an equilibrium solution of the limiting difference equation (118), then every solution of the difference equation (116) satisfies (113).

### 4.3 Single species several generation models

I start with an example of cooperative system which is feasible mathematical model in biology, that illustrates Theorems 40, 41, 42 and Corollary 5. This system can be considered as Leslie two generation population model, where each generation helps the other.

**Example 12** Consider the cooperative system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots, \quad (120)$$

where  $a, b, c, d > 0$ ,  $x_0, y_0 \geq 0$ . The equilibrium solutions  $(x, y)$  satisfy equation

$$\begin{bmatrix} (1-a)x \\ (1-d)y \end{bmatrix} = \begin{bmatrix} \frac{b}{1+y} \\ \frac{c}{1+x} \end{bmatrix}, \quad (121)$$

which implies that system (120) has always the zero equilibrium  $E_0(0, 0)$  and if it has a positive equilibrium solutions  $E_+(\bar{x}, \bar{y})$  then it is necessarily  $a < 1, d < 1$ , in which case there is the unique equilibrium solution given as

$$\bar{x} = \frac{b}{1-a} \frac{\bar{y}}{1+\bar{y}}, \quad \bar{y} = \frac{bc - (1-d)(1-a)}{(1-d)(b+1-a)},$$

when

$$(1-a)(1-d) < bc. \quad (122)$$

The Jacobian matrix of the map  $T$  associated with system (120) is

$$J_T = \begin{bmatrix} a & \frac{b}{(1+y)^2} \\ \frac{c}{(1+x)^2} & d \end{bmatrix}.$$

Thus the Jacobian matrix of the map  $T$  at the zero equilibrium  $E_0(0, 0)$  is

$$J_T(E_0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and at the positive equilibrium  $E_+(\bar{x}, \bar{y})$  is

$$J_T(E_+) = \begin{bmatrix} a & \frac{b}{(1+\bar{y})^2} \\ \frac{c}{(1+\bar{x})^2} & d \end{bmatrix} = \begin{bmatrix} a & \frac{(1-a)^2}{b} \frac{\bar{x}^2}{\bar{y}^2} \\ \frac{(1-d)^2}{c} \frac{\bar{y}^2}{\bar{x}^2} & d \end{bmatrix}.$$

The local stability of system (120) is described with the following result

**Claim 3** *Consider system (120).*

- 1.) *The positive equilibrium  $E_+(\bar{x}, \bar{y})$  of system (120) is locally asymptotically stable when (122) holds.*
- 2.) *The zero equilibrium  $E_0(0, 0)$  of system (120) is locally asymptotically stable if  $bc < (1-a)(1-d)$ ; it is a saddle point if  $bc > (1-a)(1-d)$ ; it is a nonhyperbolic equilibrium if  $bc = (1-a)(1-d)$ .*

**Proof.**

- 1.) After simplification the characteristic equation of  $J_T(E_+)$  becomes

$$\lambda^2 - (a+d)\lambda + ad - \frac{(1-a)^2(1-d)^2}{bc} = 0. \quad (123)$$

In view of Theorem 1.1.1 in [54] or Theorem 2.13 in [60]  $E_+$  is locally asymptotically stable if

$$a + d < 1 + ad - \frac{(1-a)^2(1-d)^2}{bc} < 2$$

holds. The inequality  $a + d < 1 + ad - \frac{(1-a)^2(1-d)^2}{bc}$  is equivalent to (122) and the inequality  $1 + ad - \frac{(1-a)^2(1-d)^2}{bc} < 2$  is equivalent to  $ad - 1 < \frac{(1-a)^2(1-d)^2}{bc}$ , which, in view of  $a < 1, d < 1$  is always satisfied.

2.) The characteristic equation of  $J_T(E_0)$  becomes

$$\lambda^2 - (a + d)\lambda + ad - bc = 0. \quad (124)$$

In view of Theorem 2.13 in [60]  $E_0$  is locally asymptotically stable if  $bc < (1 - a)(1 - d)$  and it is a saddle point if  $bc > (1 - a)(1 - d)$ . Finally, if  $bc = (1 - a)(1 - d)$  then  $E_0$  is a nonhyperbolic equilibrium point ( $\lambda_1 = 1$ ) of stable type ( $\lambda_2 \in (-1, 1)$ ) if  $a + d < 2$ , of unstable type  $\lambda_2 > 1$  if  $a + d > 2$ , and of resonance type  $1 - 1$  if  $a + d = 2$ .

□

By using Theorems 40, 41, 42, 45 and Corollary 5 I can formulate the following result which describe the global dynamics of system (120).

**Theorem 47** *Consider system (120).*

- 1.) *If  $a \geq 1$  then  $\lim_{n \rightarrow \infty} x_n = \infty$  and  $\lim_{n \rightarrow \infty} y_n = \infty$  if  $d \geq 1$  and  $\lim_{n \rightarrow \infty} y_n = \frac{c}{1-d}$ , if  $d < 1$ .*
- 2.) *If  $d \geq 1$  then  $\lim_{n \rightarrow \infty} y_n = \infty$  and  $\lim_{n \rightarrow \infty} x_n = \infty$  if  $a \geq 1$  and  $\lim_{n \rightarrow \infty} x_n = \frac{b}{1-a}$ , if  $a < 1$ .*
- 3.) *The positive equilibrium  $E_+(\bar{x}, \bar{y})$  of system (120) is globally asymptotically stable when (122) holds.*
- 4.) *The zero equilibrium  $E_0(0, 0)$  of system (120) is globally asymptotically stable when  $a < 1, d < 1$  and*

$$bc \leq (1 - a)(1 - d) \quad (125)$$

*holds.*

**Proof.**

- 1.) If  $a \geq 1$  then the first equation of system (120) implies  $x_{n+1} > ax_n \geq x_n$ , which shows that  $\{x_n\}_{n=1}^{\infty}$  is an increasing sequence and because there is no positive equilibrium in this case I have that  $\lim_{n \rightarrow \infty} x_n = \infty$ . In view of Theorem 45  $\{y_n\}_{n=1}^{\infty}$  is converging to the asymptotic solution of the limiting equation

$$y_{n+1} = c + dy_n, \quad n = 1, 2, \dots$$

which completes the proof in this case.

- 2.) The proof in this case is similar to the proof of case 1.) and is omitted.
- 3.) Assume that (122) holds. In view of Claim 3  $E_0$  is a saddle point and  $E_+$  is locally asymptotically stable. By using Corollary 5 I conclude that the interior of ordered interval  $\llbracket E_0, E_+ \rrbracket$  is attracted to  $E_+$ . Furthermore, any solution of system (120) different from  $E_0$  which starts on coordinate axis in one step enters the interior of ordered interval  $\llbracket E_0, E_+ \rrbracket$  and so converges to  $E_+$ . Every solution of system (120) satisfies

$$x_{n+1} \leq ax_n + b, \quad y_{n+1} \leq c + dy_n, \quad n = 0, 1, \dots,$$

which, in view of Theorem 42 means that

$$x_n \leq u_n, \quad y_n \leq v_n, \quad n = 0, 1, \dots,$$

where  $\{u_n\}, \{v_n\}$  satisfy

$$u_{n+1} = ax_n + b, \quad v_{n+1} = c + dv_n, \quad n = 0, 1, \dots$$

Since,  $a < 1, d < 1$  I obtain that

$$x_n \leq \frac{b}{1-a} + \varepsilon_0 = U_x, \quad y_n \leq \frac{c}{1-d} + \varepsilon_0 = U_y, \quad (126)$$

for some  $\varepsilon_0 > 0$  and  $n \geq N(\varepsilon_0)$ . In view of iii. of Theorem 40 every solution which starts in the interior of ordered interval  $\llbracket E_+, (U_x, U_y) \rrbracket$  is attracted



to  $E_+$ . Since the system (120) is strictly cooperative I conclude that the whole ordered interval  $\llbracket E_+, (U_x, U_y) \rrbracket$  is attracted to  $E_+$ . If a solution starts at  $(x_0, y_0)$  in the complement of  $\llbracket E_0, E_+ \rrbracket \cup \llbracket E_+, (U_x, U_y) \rrbracket$  one can find the points  $(x_l, y_l) \in \llbracket E_0, E_+ \rrbracket$  and  $(x_u, y_u) \in \llbracket E_+, (U_x, U_y) \rrbracket$  such that  $(x_l, y_l) \preceq_{ne} (x_0, y_0) \preceq_{ne} (x_u, y_u)$ . Since  $T$  is monotone map I have that  $T^n(x_l, y_l) \preceq_{ne} T^n(x_0, y_0) \preceq_{ne} T^n(x_u, y_u)$  for  $n \geq 1$ , which implies that  $\lim_{n \rightarrow \infty} T^n(x_0, y_0) = E_+$  because  $\lim_{n \rightarrow \infty} T^n(x_l, y_l) = \lim_{n \rightarrow \infty} T^n(x_u, y_u) = E_+$ .

- 4.) In view of the condition (125) system (120) has only the zero equilibrium. By using (126) I have that the map  $T$  associated with the system (120) has an invariant rectangle  $\llbracket E_0, (U_x, U_y) \rrbracket$ , with the unique equilibrium point  $E_0$  and by Theorem 40 every solution which starts in this rectangle must converge to  $E_0$ . The fact that the rectangle  $\llbracket E_0, (U_x, U_y) \rrbracket$  is also attractive completes the proof.

□

The following difference equation is known as density dependent Leslie matrix model with two age classes, juveniles and adults:

$$x_{n+1} = \frac{a_1 x_n}{1 + b_1 x_n} + \frac{a_2 x_{n-1}}{1 + b_2 x_{n-1}}, \quad n = 0, 1, \dots \quad (127)$$

where the parameters  $a_1, a_2, b_1, b_2$  are positive real numbers and the initial conditions  $x_{-1}$  and  $x_0$  are nonnegative real numbers. Here  $x_n$  is the size of population at  $n$ -th generation. This model was considered first by Kulenović and Yakubu [74] in 2004 and later by Franke and Yakubu [28]-[33], where the extensions of this model to the periodic environment are considered.

**Theorem 48** *The density dependent Leslie matrix model (127) exhibits the following properties:*

1. Equation (127) has always the zero equilibrium and when  $a_1 + a_2 > 1$  has the unique positive equilibrium  $\bar{x}$ .
2. Assume  $a_1 + a_2 < 1$ . Then the zero equilibrium of Equation (127) is globally asymptotically stable.
3. Assume  $a_1 + a_2 > 1$ . Then the unique positive equilibrium  $\bar{x}$  of Equation (127) is globally asymptotically stable.
4. Assume that  $a_1 + a_2 > 1$  holds. All solutions of Equation (127) satisfy

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \bar{x}}{x_n - \bar{x}} = \lambda_{\pm}(\bar{x}), \quad (128)$$

where  $\lambda_{\pm}(\bar{x})$  are the real roots of characteristic equation at the equilibrium  $\bar{x}$ :

5. Equation (127) has both, the oscillatory and nonoscillatory solutions. The oscillatory solutions have the semicycles of length one, with the possible exception of the first semicycle.

An application of Theorems 45 and 48 yields:

**Example 13** Consider the difference equation

$$x_{n+1} = \frac{c_1(n)x_n}{1 + b_1x_n} + \frac{c_2(n)x_{n-1}}{1 + b_2x_{n-1}}, \quad n = 0, 1, \dots \quad (129)$$

where the parameters  $c_1(n), c_2(n), b_1, b_2$  are positive real sequences and numbers and the initial conditions  $x_{-1}$  and  $x_0$  are nonnegative real numbers. If

$$\lim_{n \rightarrow \infty} c_i(n) = a_i > 0, \quad i = 1, 2.$$

If either  $a_1 + a_2 < 1$  or  $a_1 + a_2 > 1$ , then the global dynamics of non-autonomous difference equation (129) is same as the global dynamics of the limiting equation Equation (127) and is described by Theorem 48.

**Remark 6** Similarly, as for the Beverton-Holt equation the difficult case is when the limiting equation (127) is non-hyperbolic. In this case the dynamics of non autonomous equation can be quite different than the dynamics of the limiting equation.

I consider the following difference equation as a generalization of density dependent Leslie matrix model with two age classes

$$x_{n+1} = \sum_{i=0}^k a_i f_i(x_{n-i}), \quad n = 0, 1, \dots \quad (130)$$

where the parameters  $a_i \geq 0, i = 0, \dots, k, \sum_{i=0}^k a_i > 0$  and  $f_i(u)$  satisfy the following conditions:

$$f_i : [0, \infty) \rightarrow [0, \infty), f_i(u) \leq u, \quad (131)$$

for all  $i = 0, 1, \dots, k$ . Examples of functions which satisfy condition (131) are

$$f(u) = u, \quad f(u) = \frac{u}{1+u}, \quad f(u) = \frac{u^2}{1+u^2}, \quad f(u) = 1 - e^{-u}.$$

Condition (131) implies that  $f_i(0) = 0, i = 0, \dots, k$  and that 0 is always an equilibrium of Equation (130). Furthermore, if there exists a positive equilibrium  $\bar{x} > 0$ , then  $\sum_{i=0}^k a_i \geq 1$ .

#### 4.4 Local and Global Dynamics of Several Generation Models

First I state and prove the local stability result which is sharp.

**Theorem 49** *The equilibrium  $\bar{x}$  of Equation (130) is one of the following:*

(a) *locally asymptotically stable if  $\sum_{i=0}^k a_i f'_i(\bar{x}) < 1$ ,*

(b) *unstable if  $\sum_{i=0}^k a_i f'_i(\bar{x}) > 1$ .*

**Proof.** This result is the consequence of Theorem 3 in Janowski and Kulenović [44] applied to the linearization

$$x_{n+1} = \sum_{i=0}^k a_i f'_i(\bar{x}) x_{n-i}, \quad n = 0, 1, \dots \quad (132)$$

at the equilibrium  $\bar{x}$ . □

The global result is simple to state and apply

**Theorem 50** *Consider Equation (130) subject to the condition (131). The zero equilibrium  $\bar{x} = 0$  of Equation (130) is globally asymptotically stable if*

$$\sum_{i=0}^k a_i < 1. \quad (133)$$

**Proof.** The result is an immediate consequence of Corollary 1. in [44] applied to the following linearization of Equation (130)

$$x_{n+1} = \sum_{i=0}^k a_i \frac{f_i(x_{n-i})}{x_{n-i}} x_{n-i} = \sum_{i=0}^k g_i x_{n-i}, \quad n = 0, 1, \dots$$

In this case

$$\sum_{i=0}^k g_i = \sum_{i=0}^k a_i \frac{f_i(x_{n-i})}{x_{n-i}} \leq \sum_{i=0}^k a_i < 1$$

and Corollary 1. in [44] imply the global asymptotic stability of the zero equilibrium. □

Assume that Equation (130) has a positive equilibrium  $\bar{x} > 0$ . Then (133) is not satisfied, that is  $\sum_{i=0}^k a_i \geq 1$ .

The global result requires an additional condition which is well known.

**Theorem 51** *Consider Equation (130) subject to the condition (131) and*

$$|f_i(u) - f_i(\bar{x})| \leq C_i |u - \bar{x}|, \quad u > 0, \quad (134)$$

where  $C_i > 0$  are constants, for all  $i = 0, \dots, k$ . The equilibrium  $\bar{x}$  of Equation (130) is globally asymptotically stable if

$$\sum_{i=0}^k a_i C_i < 1. \quad (135)$$

is satisfied.

**Proof.** If  $f_i(u)$  is differentiable on  $[0, \infty)$  then condition (134) follows from the condition

$$|f'_i(u)| \leq C_i, \quad i = 0, 1, \dots, k, \quad u > 0$$

The result is an immediate consequence of Corollary 1. in [44] applied to the following linearization of Equation (130)

$$x_{n+1} - \bar{x} = \sum_{i=0}^k a_i \frac{f_i(x_{n-i}) - f_i(\bar{x})}{x_{n-i} - \bar{x}} (x_{n-i} - \bar{x}) = \sum_{i=0}^k g_i (x_{n-i} - \bar{x}), \quad n = 0, 1, \dots,$$

which by substitution  $y_n = x_n - \bar{x}$ , becomes the linearized equation

$$y_{n+1} = \sum_{i=0}^k g_i y_{n-i}, \quad n = 0, 1, \dots,$$

where

$$g_i = a_i \frac{f_i(x_{n-i}) - f_i(\bar{x})}{x_{n-i} - \bar{x}}, \quad i = 0, \dots, k.$$

□

By using the monotone convergence results in [54, 60, 63] I obtain the following powerful global asymptotic stability result

**Theorem 52** *Consider Equation (130) subject to the condition*

$$f_i(u) \leq U_i, \quad u \geq 0, i = 0, \dots, k, \quad (136)$$

where  $f_i(u)$  is nondecreasing for every  $u$ . If there exists a constant  $L > 0$  such that

$$\sum_{i=0}^k a_i \geq \frac{L}{\min_{i=0, \dots, k} f_i(L)}, \quad (137)$$

then, if Equation (130) has the unique positive equilibrium  $\bar{x}$ , it is globally asymptotically stable.

**Proof.** Set

$$F(u_0, u_1, \dots, u_k) = \sum_{i=0}^k a_i f_i(u_i), \quad u_i \geq 0, i = 0, \dots, k.$$

Then  $F(u_0, u_1, \dots, u_k) \leq \sum_{i=0}^k a_i U_i = U$ . Furthermore, if there exists a constant  $L > 0$  such that  $u_i \geq L$ ,  $i = 0, \dots, k$ , then in view of (137) I would have

$$F(u_0, u_1, \dots, u_k) = \sum_{i=0}^k a_i f_i(u_i) \geq \sum_{i=0}^k a_i f_i(L) \geq \min_{i=0, \dots, k} f_i(L) \sum_{i=0}^k a_i \geq L.$$

This shows that the interval  $[L, U]$  is an invariant interval for the function  $F(u_0, u_1, \dots, u_k)$ , which is nondecreasing in all its arguments. In view of Theorem [54, 63], the fact that Equation (130) has the unique positive equilibrium  $\bar{x}$  implies that this equilibrium is globally asymptotically stable.  $\square$

An application of Theorems 1-4 gives the following result for Leslie or Beverton-Holt model with  $k$  generations :

**Theorem 53** *The Beverton-Holt model with  $k$  generations*

$$x_{n+1} = \sum_{i=0}^k a_i \frac{x_{n-i}}{1 + x_{n-i}}, \quad n = 0, 1, \dots, a_i \geq 0, \sum_{i=0}^k a_i > 0, \quad (138)$$

*has the following properties:*

- (a) *If  $\sum_{i=0}^k a_i < 1$ , then the zero equilibrium is globally asymptotically stable.*
- (b) *If  $\sum_{i=0}^k a_i = 1$ , then the zero equilibrium is the global attractor.*
- (c) *If  $\sum_{i=0}^k a_i > 1$ , then the positive equilibrium  $\bar{x} = \sum_{i=0}^k a_i - 1$  is globally asymptotically stable.*

Now (a) and (b) follow from Theorem 50 and the fact that

$$\sum_{i=0}^k a_i \frac{x_{n-i}}{1 + x_{n-i}} < \sum_{i=0}^k a_i < 1, \quad n = 0, 1, \dots$$

In the case of (c), the global asymptotic stability of the positive equilibrium  $\bar{x}$  follows from Theorem 52 with  $L$  and  $U$  as:  $U = \sum_{i=0}^k a_i$  and  $0 < L \leq \bar{x}$ .

**Theorem 54** *The sigmoid Beverton-Holt model with  $k$  generations which exhibits the Allee's effect:*

$$x_{n+1} = \sum_{i=0}^k a_i \frac{x_{n-i}^2}{1 + x_{n-i}^2}, \quad n = 0, 1, \dots; \quad a_i \geq 0, \sum_{i=0}^k a_i > 0, \quad (139)$$

*has the following properties:*

- (a) *If  $\sum_{i=0}^k a_i < 2$ , then the zero equilibrium is globally asymptotically stable.*
- (b) *If  $\sum_{i=0}^k a_i = 2$ , then the zero equilibrium is globally asymptotically stable within its basin of attraction which contains  $[0, 1)^{k+1}$ . The positive equilibrium  $\bar{x} = 1$  is locally non-hyperbolic and is global attractor within its basin of attraction which contains  $[1, \infty)^{k+1}$ .*
- (c) *If  $\sum_{i=0}^k a_i > 2$ , then the zero equilibrium is globally asymptotically stable within its basin of attraction which contains  $[0, \bar{x}_-)^{k+1}$ . The larger positive equilibrium  $\bar{x}_+$  is globally asymptotically stable within its basin of attraction which contains  $(\bar{x}_-, \infty)^{k+1}$ .*

**Proof.** The linearized equation of Equation (139) at an equilibrium point  $\bar{x}$  is

$$x_{n+1} = \sum_{i=0}^k 2a_i \frac{\bar{x}}{(1 + \bar{x}^2)^2} x_{n-i}, \quad n = 0, 1, \dots \quad (140)$$

and the characteristic equation of Equation (140) is:

$$\lambda^{k+1} = \sum_{i=0}^k 2a_i \frac{\bar{x}}{(1 + \bar{x}^2)^2} \lambda^{k-i}. \quad (141)$$

Theorem 49 implies the local stability of the zero and the positive equilibrium points. Observe that the equilibrium equation of Equation (139) can have at most two positive solutions.

Every solution of Equation (139) satisfies  $x_n \leq \sum_{i=0}^k a_i = U, n = k + 1, k + 2, \dots$ . Using this it follows that the interval  $[\mathbf{x}_-, \mathbf{U}]$ , where  $\mathbf{U} = (U, U, \dots, U)$  is

invariant set for monotone map  $T$ , which contains the unique fixed point  $\mathbf{x}_-$ . In view of Theorem 40 every orbit of  $T$  converges to  $\mathbf{x}_-$ , which means that  $\lim x_n = x_-$  as  $n \rightarrow \infty$ .

Assume that  $\sum_{i=0}^k a_i > 2$ . Then Equation (139) has two positive equilibrium solutions  $x_- < x_+$ , where  $x_-$  is unstable and  $x_+$  is asymptotically stable. In a similar way as above the interior of the ordered interval  $[[0, \mathbf{x}_-]]$  is a subset of the basin of attraction  $\mathcal{B}(\mathbf{0})$ , and the union of the interiors of the ordered intervals  $[[\mathbf{x}_-, \mathbf{x}_+]]$  and  $[[\mathbf{x}_+, \mathbf{U}]]$  is a subset of the basin of attraction  $\mathcal{B}(\mathbf{x}_+)$ .

When  $\sum_{i=0}^k a_i < 2$  Equation (139) has only the zero equilibrium and can be linearized as

$$x_{n+1} = \sum_{i=0}^k a_i \frac{x_{n-i}}{1 + x_{n-i}^2} x_{n-i} = \sum_{i=0}^k g_i x_{n-i}, \quad n = 0, 1, \dots; \quad (142)$$

Since  $\frac{u}{1+u^2} \leq 1/2$  for every  $u$  I have

$$\sum_{i=0}^k g_i \leq \sum_{i=0}^k \frac{a_i}{2} < 1, \quad n = 0, 1, \dots$$

which by Theorem 2 in [44] or Theorem 51 implies that the zero equilibrium is globally asymptotically stable.

The positive equilibrium  $\bar{x}$  satisfies the equation

$$x^2 - \sum_{i=0}^k a_i x + 1 = 0, \quad (143)$$

which either has one positive solution  $\bar{x} = 1$  when  $\sum_{i=0}^k a_i = 2$  or it has two positive solutions when  $\sum_{i=0}^k a_i > 2$ .

In the case when  $\sum_{i=0}^k a_i = 2$ , the characteristic equation (141) takes the form

$$\lambda^{k+1} = \sum_{i=0}^k \frac{a_i}{2} \lambda^{k-i},$$

with  $\lambda = 1$  as a solution, which shows that  $\bar{x} = 1$  is stable and non-hyperbolic equilibrium.



In the case when  $\sum_{i=0}^k a_i > 2$ , I have two positive equilibrium solutions  $\bar{x}_{\pm}$  which satisfy Equation (143) and so  $\bar{x}_- < 1 < \bar{x}_+$ . In view of Theorem 3 in [44]  $\bar{x}$  is locally asymptotically stable if and only if

$$\sum_{i=0}^k 2a_i \frac{\bar{x}}{(1 + \bar{x}^2)^2} < 1$$

which by Equation (143) implies

$$2 < \bar{x} \sum_{i=0}^k a_i = \bar{x}^2 + 1 \Leftrightarrow \bar{x} > 1.$$

Similarly one can show that the equilibrium  $\bar{x} > 1$  is unstable if and only if  $\bar{x} < 1$ . Consequently,  $\bar{x}_+$  is locally asymptotically stable and  $\bar{x}_-$  is unstable.  $\square$

In the special case  $k = 2$ , I obtain more precise description of the basins of attraction of the equilibrium points as well as the period-two solutions. The Leslie model with 2 generations which exhibits the Allee's effect may possess up to three minimal period-two solutions which in certain cases may have substantial basins of attraction as it was shown in [7]. I summarize the results in [7] as follows:

**Theorem 55** *Consider the difference equation*

$$x_{n+1} = a_0 \frac{x_n^2}{1 + x_n^2} + a_1 \frac{x_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \dots; \quad (144)$$

where  $a_0, a_1 \geq 0, a_0 + a_1 > 0$ , and assume that it does not have period-two solutions. Then

- (a) *If  $a_0 + a_1 < 2$ , then the zero equilibrium of Equation (144) is globally asymptotically stable.*
- (b) *If  $a_0 + a_1 = 2$ , then the zero equilibrium of Equation (144) is globally asymptotically stable within its basin of attraction which is the region below the global stable manifold  $\mathcal{W}^s(E), E(1, 1)$  in the North-east ordering. The positive equilibrium  $\bar{x} = 1$  is locally non-hyperbolic and is global attractor within*

its basin of attraction, which is the region above the global stable manifold  $\mathcal{W}^s(E)$  in the North-east ordering.

(c) If  $a_0 + a_1 > 2$ , then Equation (144) has three equilibrium points  $x_0 = 0 < \bar{x}_- < \bar{x}_+$  and zero, one, two or three minimal period-two solutions. If there is no minimal period-two solutions and the equilibrium point  $E_-$  is a saddle point, then the zero equilibrium is globally asymptotically stable within its basin of attraction which is the region below the global stable manifold  $\mathcal{W}^s(E_-)$ ,  $E_- = (\bar{x}_-, \bar{x}_-)$  in the North-east ordering. The larger positive equilibrium  $\bar{x}_+$  is the global attractor within its basin of attraction, which is the region above the global stable manifold  $\mathcal{W}^s(E_-)$  in the North-east ordering.

(d) If  $a_0 + a_1 > 2$ , and Equation (144) has one minimal period-two solution  $\{\Phi, \Psi\}$  which is a saddle point then the boundary between the basins of attraction of  $E_0(x_0, x_0)$  and  $E_+ = (\bar{x}_+, \bar{x}_+)$  is the union of the stable manifolds of  $E_-$  and the stable manifolds of the period-two points  $\mathcal{W}^s(P)$  and  $\mathcal{W}^s(Q)$ ,  $P = (\Phi, \Psi)$ ,  $Q = (\Phi, \Psi)$ .

(e) If  $a_0 + a_1 > 2$ , and Equation (144) has two minimal period-two solutions  $\{\Phi_i, \Psi_i\}$ ,  $i = 1, 2$ , where one of them is a saddle point and the other one is non-hyperbolic solution of the stable type, then the boundary between the basins of attraction of  $E_0(x_0, x_0)$  and  $E_+ = (\bar{x}_+, \bar{x}_+)$  is the union of the stable manifolds of  $E_-$  and the period-two points  $\mathcal{W}^s(P)$  and  $\mathcal{W}^s(Q)$ ,  $P = (\Phi, \Psi)$ ,  $Q = (\Phi, \Psi)$ . Furthermore, the basin of the attraction of the non-hyperbolic period-two solution is the region between the stable manifolds of two period-two solutions  $\{\Phi_i, \Psi_i\}$ ,  $i = 1, 2$ .

(f) If  $a_0 + a_1 > 2$ , and Equation (144) has three minimal period-two solutions  $\{\Phi_i, \Psi_i\}$ ,  $i = 1, 2, 3$ , such that  $\{\Phi_1, \Psi_1\} \preceq_{ne} \{\Phi_2, \Psi_2\} \preceq_{ne} \{\Phi_3, \Psi_3\}$  where

$\{\Phi_i, \Psi_i\}, i = 1, 3$  are saddle points and  $\{\Phi_2, \Psi_2\}$  is locally asymptotically stable, then the boundary of the basins of attraction of  $E_0(x_0, x_0)$  is the stable manifold of  $\{\Phi_1, \Psi_1\}$ , the boundary of the basins of attraction of  $E_+$  is the stable manifold of  $\{\Phi_3, \Psi_3\}$ , and the basin of the attraction of the period-two solution  $\{\Phi_2, \Psi_2\}$  is the region between the stable manifolds of two period-two solutions  $\{\Phi_i, \Psi_i\}, i = 1, 3$ .

See Figure 1. for visual interpretation of Cases (d) and (f). See Figure 2. for visual interpretation of Case (e).

**Proof.** The linearized equation of Equation (144) at an equilibrium point  $\bar{x}$  is

$$x_{n+1} = 2a_0 \frac{\bar{x}}{(1 + \bar{x}^2)^2} x_n + 2a_1 \frac{\bar{x}}{(1 + \bar{x}^2)^2} x_{n-1}, \quad n = 0, 1, \dots \quad (145)$$

and the characteristic equation of Equation (145) is:

$$\lambda^2 = 2a_0 \frac{\bar{x}}{(1 + \bar{x}^2)^2} \lambda + 2a_1 \frac{\bar{x}}{(1 + \bar{x}^2)^2}. \quad (146)$$

The linearized equation (146) at the zero equilibrium is  $\lambda^2 = 0$ , which shows that the zero equilibrium is locally asymptotically stable. The linearized equation (146) at the equilibrium  $\bar{x} = 1$  is  $\lambda^2 = \frac{1}{2}\lambda + \frac{1}{2}$ , with the eigenvalues  $\lambda_1 = 1, \lambda_2 = \frac{a_0 - 2}{2} \in (-1, 0)$ , which shows that  $\bar{x} = 1$  is the nonhyperbolic equilibrium of stable type. The necessary and sufficient condition for the equilibrium  $\bar{x} = 1$  to be nonhyperbolic is  $a_0 + a_1 = 2$ . The necessary and sufficient condition for Equation (144) to have three equilibrium points  $\bar{x}_0 = 0, \bar{x}_- = \frac{1}{2}(a_0 + a_1 - \sqrt{(a_0 + a_1)^2 - 4}), \bar{x}_+ = \frac{1}{2}(a_0 + a_1 + \sqrt{(a_0 + a_1)^2 - 4})$  is  $a_0 + a_1 > 2$ . When  $a_0 + a_1 > 2$  the direct calculation shows that  $\bar{x}_-$  is a saddle point and  $\bar{x}_+$  is locally asymptotically stable.

As a consequence of the results in [6] every solution  $\{x_n\}$  of Equation (144) has two eventually monotone subsequences  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$ , which in view of the

boundedness of solutions of Equation (144) implies that all solutions of Equation (144) converge either to the equilibrium or to period-two solutions.

When  $a_0 + a_1 < 2$  Equation (144) has only the zero equilibrium and can be linearized as

$$x_{n+1} = a_0 \frac{x_n}{1 + x_n^2} x_n + a_1 \frac{x_{n-1}}{1 + x_{n-1}^2} x_{n-1} = g_0 x_n + g_1 x_{n-1}, \quad n = 0, 1, \dots; \quad (147)$$

Since  $\frac{u}{1+u^2} \leq 1/2$  for every  $u$  I have

$$g_0 + g_1 = a_0 \frac{x_n}{1 + x_n^2} + a_1 \frac{x_{n-1}}{1 + x_{n-1}^2} \leq \frac{a_0}{2} + \frac{a_1}{2} < 1, \quad n = 0, 1, \dots$$

which by Theorem 2 in [44] or Theorem 51 implies that the zero equilibrium is globally asymptotically stable.

The proof of global results in cases (b) and (c) follows from the results for monotone systems of difference equations in [67, 68].

In the case  $a_0 + a_1 = 2$ , Theorem 1 of [68] implies the existence of the stable manifold which is passing through the equilibrium point  $E(1, 1)$  and/or period-two solution in the phase plane  $(x_{-1}, x_0)$ . Furthermore, Theorems 4 and 6 of [68] can be used to obtain the global behavior of the solutions of Equation (144). See [68] for details.

Similarly, in the case  $a_0 + a_1 > 2$ , Theorems 1,2,4,5 of [68] guarantee the existence of the stable manifold and unstable manifolds with described properties, which are passing through the equilibrium points  $E_{-}(x_{-}, x_{-})$  and the period-two solutions in the phase plane  $(x_{-1}, x_0)$ . The global behavior of the solutions of Equation (144) follows from Theorems 4,5 of [68] and is explained in great details in [7].  $\square$

**Remark 7** The algebraic conditions on the coefficients  $a_0, a_1$  for having 1, 2 or 3 period-two solutions for model (139) are given in [7]. I have also shown in [7] that

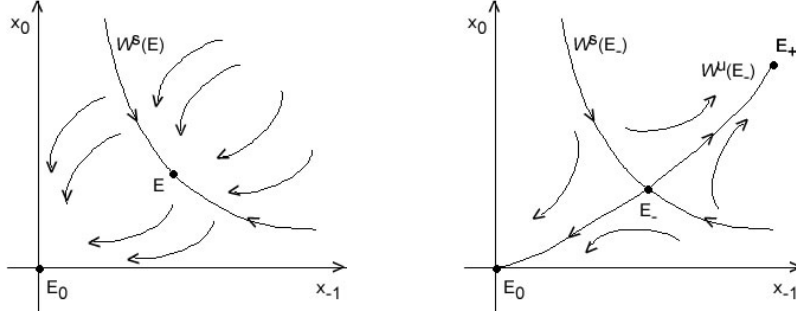


Figure 10. Global Dynamics of two generations Leslie model with the Allee's effect: Cases (b) and (c).

another feasible model

$$x_{n+1} = Ax_n + \frac{Bx_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \dots \quad (148)$$

where  $A, B > 0$  and the initial conditions  $x_{-1}, x_0$  are non-negative, have similar dynamic scenarios as well as the combination of Beverton-Holt and sigmoid Beverton-Holt mode:

$$x_{n+1} = A \frac{x_n}{1 + x_n} + \frac{Bx_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 0, 1, \dots \quad (149)$$

where  $A, B > 0$  and the initial conditions  $x_{-1}, x_0$  are non-negative. In both cases the conditions for existence of period-two solutions are rather complicated.

The biological implications of models (139), (148) and (149) is that for some values of parameters of these models period-two behavior emerges with substantial basin of attraction.

#### 4.5 Equations with Nonconstant Coefficients: Asymptotic constant case

I consider the nonautonomous difference equation (116) as a generalization of Equation (130) where the sequences  $a_i(n) \geq 0, i = 0, \dots, k, \sum_{i=0}^k a_i(n) > 0, n = 0, 1, \dots$  and  $f_i(u)$  satisfy (131). The global attractivity and global stability results in [44] can be applied here to give some robust global stability results:

**Theorem 56** Consider Equation (116) subject to (131). If  $a_i(n)$  are bounded sequences for  $i = 0, \dots, k$  and

$$(i) \quad \sum_{i=0}^k \max_{n=0,1,\dots} \{a_i(n)\} \leq a < 1, \quad (150)$$

for some constant  $a > 0$ , then the zero equilibrium is globally asymptotically stable.

$$(ii) \quad \sum_{i=0}^k \max_{n=0,1,\dots} \{a_i(n)\} \leq 1, \quad (151)$$

then the zero equilibrium is stable.

An application of Theorem 46 and the global attractivity result proved in [74] for constant coefficient case gives the following global attractivity result for asymptotically constant  $k$ -stage Beverton-Holt model

$$x_{n+1} = \sum_{i=0}^{k-1} a_i(n) \frac{x_{n-i}}{1 + x_{n-i}}, \quad x_{-k+1}, \dots, x_0 \geq 0, \quad a_i(n) \geq 0, \quad i = 0, \dots, k-1; n = 0, 1, \dots \quad (152)$$

**Example 14** Consider the difference equation (152) where (117) holds. Then the following result holds

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{k-1} a_i(n) = a \begin{cases} < 1 \\ > 1 \end{cases} \implies \lim_{n \rightarrow \infty} x_n = \begin{cases} 0, & \text{if } x_{1-k}, \dots, x_0 \geq 0 \\ a - 1, & \text{if } x_{1-k}, \dots, x_0 > 0. \end{cases}$$

An application of Theorems 40, 46 and Corollary 5 gives the following global attractivity result for asymptotically constant  $k$ -stage Beverton-Holt model with the Allee effect

$$x_{n+1} = \sum_{i=0}^{k-1} a_i(n) \frac{x_{n-i}^2}{1 + x_{n-i}^2}, \quad x_{-k+1}, \dots, x_0 \geq 0, \quad a_i(n) \geq 0, \quad i = 0, \dots, k-1; n = 0, 1, \dots \quad (153)$$

**Example 15** Consider the difference equation (153) where (117) holds. Then the following result holds

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{k-1} a_i(n) = a \begin{cases} < 2 \\ > 2 \end{cases} \implies \lim_{n \rightarrow \infty} x_n = \begin{cases} 0, & \text{if } x_m \in [0, \bar{x}_-), m = 1 - k, \dots, 0 \\ \bar{x}_+, & \text{if } x_m \in (\bar{x}_-, \infty), m = 1 - k, \dots, 0, \end{cases}$$

where  $\bar{x}_-$  is the smaller and  $\bar{x}_+$  is the bigger positive equilibrium.

It should be noticed that Examples 14 and 15 do not cover a nonhyperbolic case, which is the case when  $\lim_{n \rightarrow \infty} \sum_{i=0}^{k-1} a_i(n) = 1$  and  $\lim_{n \rightarrow \infty} \sum_{i=0}^{k-1} a_i(n) = 2$  respectively.

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